On the tree-width of even-hole-free graphs

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Motivation

Even-hole-free graphs naturally arise in the context of perfect graphs.

Better understanding of the structure of even-hole-free graphs

$\rightsquigarrow$ efficent algorithms for:

- Computational problems on (subclasses of) even-hole-free graphs
  Open: Colouring, Stable Set in PTIME?
- Testing even-hole-freeness in the bounded-degree model of Property Testing
  ‘Approximate recognition’ in sublinear time

Contents

Preliminaries

- All graphs are simple, undirected and finite.
- All graph classes are closed under isomorphism, and a graph class is sometimes called a property.
- Class $\mathcal{C}$ of graphs has bounded degree, if there is a constant $d \in \mathbb{N}$ such that all graphs in $\mathcal{C}$ have degree $\leq d$.

K_n denotes the complete graph on n vertices.

C_n denotes the cycle of length n ($n \geq 3$).

Grids and walls

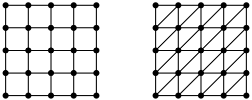


Figure: (5×5)-grid; Triangulated (5×5)-grid

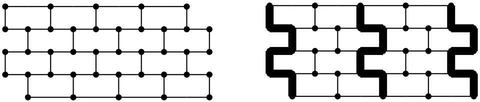


Figure: Elementary (5×5)-wall; Three ‘vertical’ paths highlighted

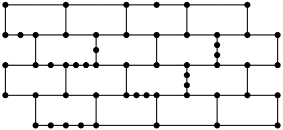


Figure: (5×5)-wall.

Subgraphs, contractions and minors

For graphs G and H:

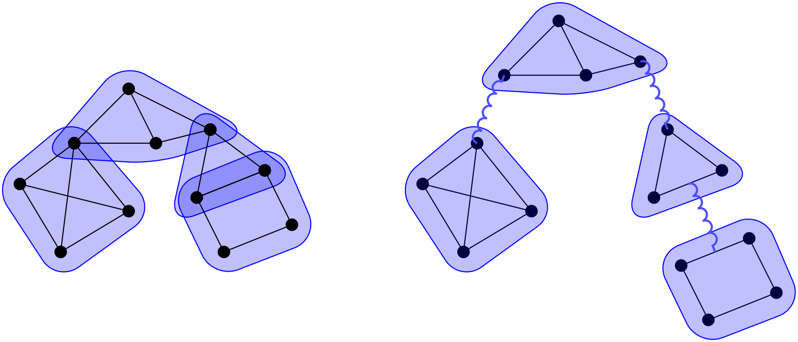
- H is an induced subgraph of G, if H can be obtained from G by vertex deletions.
- H is a subgraph of G, if H can be obtained from G by vertex and/or edge deletions.
- H is a contraction of G, if H can be obtained from G by edge contractions.
- H is a minor of G, if H is a contraction of a subgraph of G.

We say that G is H-free, if G does not contain H as induced subgraph.

Tree-width (Intuitively)

Tree-width measures how close a graph is to being a tree.

G has tree-width $\leq k$, if G can be pieced together from subgraphs of size $\leq k+1$ in a tree-like fashion:



Tree-width (Definition)

Tree decomposition (T, B) of G:

- Tree T
- A family $B = (B_t)_{t \in V(T)}$ with $B_t \subseteq V(G)$ (*bags*)

such that:

(1) $v \in V(G) \Rightarrow v \in B_t$ for some $t \in V(T)$

(2) $\{u, v\} \in E(G) \Rightarrow \{u, v\} \subseteq B_t$ for some $t \in V(T)$

(3) For every $v \in V(G)$ the set $\{t \in V(T) \mid v \in B_t\}$ is connected in T

Width of a tree decomposition:

$$\max \left\{ |B_t| \ : \ t \in V(T) \right\} - 1$$

Tree-width of G:

$\mathrm{tw}(G) =$ Minimum width over all tree decompositions of G

- Introduced in [N. Robertson, P. D. Seymour. Graph Minors. II, 1986.]

Tree-width (Examples)

A graph class $\mathcal{C}$ has bounded tree-width, if there exists a $t \in \mathbb{N}$ such that all members of $\mathcal{C}$ have tree-width at most t.
Otherwise, $\mathcal{C}$ has unbounded tree-width.

Examples

- $\text{tw}(\textit{trees}) \leq 1$,
- $\text{tw}(K_n) = n - 1$,
- $\text{tw}\big((n \times n)\text{-}\textit{grid}\big) = n$,
- *Walls have unbounded tree-width.*
- *H subgraph of G* $\Rightarrow \text{tw}(H) \leq \text{tw}(G)$.
  Similarly, if H is induced subgraph or contraction or minor of G.

Algorithmic use of tree-width

Many problems that are NP-hard in general become tractable on bounded tree-width.

Theorem (B. Courcelle 1990)

Let $t \in \mathbb{N}$, and $\mathcal{C}$ be a class of graphs of tree-width $\leq t$.
Every property expressible in monadic second-order logic with counting (CMSO) is decidable in linear running time on $\mathcal{C}$.

Examples

Expressible in CMSO:

- *stable set, clique, vertex cover, dominating set,*
- *(non-)existence of a fixed (induced) subgraph H*
- *planarity, bounded genus, excluded minor*
- *connectivity, colorability, Hamiltonicity,*
- *even number of vertices, perfectness, even-hole-freeness*

Even-hole-free graphs

Let C be a cycle in G. An edge $e \in E(G)$ is a chord of C, if the endpoints of e are vertices of C that are not adjacent on C.
A hole in a graph is a chordless cycle of length at least 4. It is even or odd according to the parity of its length.
A graph is even-hole-free (ehf) if it does not contain an even hole.

Examples

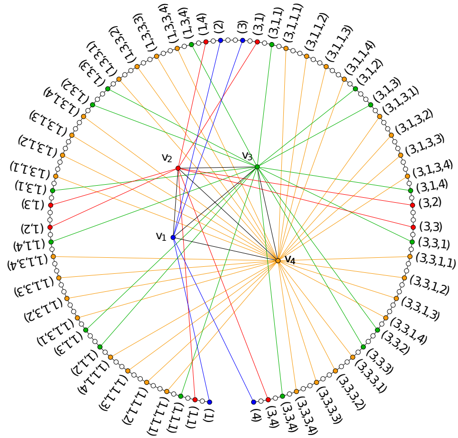
Complete graphs, trees, chordal (i. e. hole-free) graphs are ehf.
Thetas and prisms are not ehf:

Figure: Theta and prism

Remark

Complete graphs are ehf $\Rightarrow$ *ehf graphs have unbounded tree-width.*

An even-hole-free graph



[On rank-width of even-hole-free graphs, I. A., N.-K. Le, H. Müller, M. Radovanovic, N. Trotignon, K. Vušković, 2017]

Which ehf graphs have bounded tw?

Theorem (A. Silva, A. A. da Silva, C. Linhares Sales, 2010)
Planar ehf graphs have bounded tree-width.

Theorem (K. Cameron, M. da Silva, S. Huang, K. Vušković, 2018)
(Even-hole, K_3)-free graphs have bounded tree-width.

Do ehf graphs of bounded clique number have bounded tw?
[K. Cameron, S. Chaplick, C. Hoàng, 2018]

Theorem (N. L. D. Sintiari, N. Trotignon, 2019)
No. (Even-hole, K_4)-free graphs of unbounded tree-width exist.

The construction has unbounded degree and K_n-minors for arbitrarily large n.

Question: Are these necessary?

Our contributions (minors)

Theorem (P. Aboulker, I. A., E. Kim, N. L. D. Sintiari, N. Trotignon, 2020)
Ehf graphs excluding a minor have bounded tree-width.
(theta, prism)-free graphs excluding a minor have bounded tree-width.

This implies that planar ehf graphs have bounded tree-width.

For the proof we establish an 'induced grid theorem' for graphs excluding a minor.

Our contributions (degree)

Conjecture (P. Aboulker, I. A., E. Kim, N. L. D. Sintiari, N. Trotignon, 2020)
Ehf graphs of bounded degree have bounded tree-width.

Theorem (P. Aboulker, I. A., E. Kim, N. L. D. Sintiari, N. Trotignon, 2020)

- *Subcubic ehf graphs have bounded tree-width.*
  We give a structure theorem for subcubic (theta, prism)-free graphs.
- *(Even-hole, pyramid)-free graphs of degree ≤ 4 have bounded tree-width.*
  Combines structural results to show that no K_6-minor occurs.

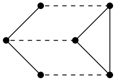


Figure: Pyramid

- Implications in Property Testing...

The conjecture is proven!

Theorem (T. Abrishami, M. Chudnovsky, K. Vušković, 2020)

Ehf graphs of bounded degree have bounded tree-width.
Even holds for C_4-free, odd-signable graphs of bounded degree.

With a theorem from [I. A., F. Harwath, 2018], it follows:

Corollary (T. Abrishami, M. Chudnovsky, K. Vušković, 2020)

Even-hole-freeness is testable in the bounded-degree model of property testing with constant query complexity and sublinear running time.

Contents

Even-hole-free graphs excluding a minor

Theorem (P. Aboulker, I. A., E. Kim, N. L. D. Sintiari, N. Trotignon, 2020)

Ehf graphs excluding a minor have bounded tree-width.
(theta, prism)-free graphs excluding a minor have bounded tree-width.

- The line graph of G is the graph $L(G)$ with $V(L(G)) = E(G)$ and two vertices in $L(G)$ are adjacent, if their corresponding edges in G share a vertex.
- Graph G is chordless, if no cycle in G has a chord.
- We call the line graph of a chordless $(k \times k)$-wall a $(k \times k)$-co-wall.

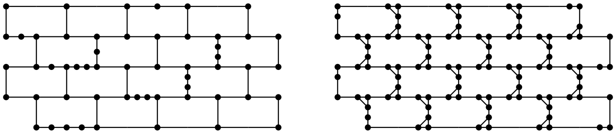


Figure: A (5×5)-wall and a (5×5)-co-wall.

‘Induced wall theorem’ for graphs excluding a minor

Theorem (P. Aboulker, I. A., E. Kim, N. L. D. Sintiari, N. Trotignon, 2020)
Given H, ex. a function f such that for every H-minor-free G and k:

- *Tree-width$(G) \leq f(k)$, or*
- *G contains a $(k \times k)$-wall or a $(k \times k)$-co-wall as induced subgraph.*

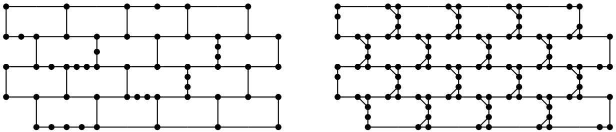


Figure: A (5×5)-wall and a (5×5)-co-wall.

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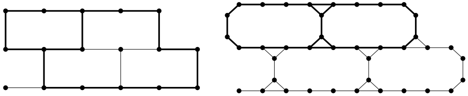
- *Tree-width$(G) \leq f(k)$, or*
- *G contains a $(k \times k)$-wall or a $(k \times k)$-co-wall as induced subgraph.*

A theta in (3×3)-wall and a prism in the (3×3)-co-wall:



Wall-tw-duality

Theorem (N. Robertson, P. D. Seymour, 1986)

Ex. a function f such that for every graph G and k:

- *Tree-width$(G) \leq f(k)$, or*
- *G contains a $(k \times k)$-wall as a subgraph.*

Note, 'subgraph' cannot be replaced by 'induced subgraph'.

Proof sketch, 1

We use:

Theorem (F. Fomin, P. Golovach, D. Thilikos, 2011)

For every H ex. a function f such that for every connected H-minor-free graph G and k:

- *Tree-width$(G) \leq f(k)$, or*
- *G contains a Γ_k or Π_k as a contraction.*

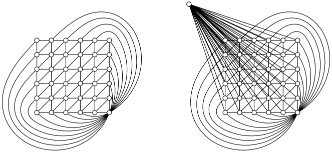
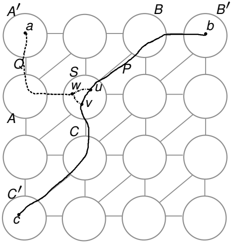


Figure: Γ_6 and Π_6

Proof sketch, 2

- Assume G is connected and H-minor-free, let k be large enough, and assume $\text{tw}(G) > f(k)$. Then G contains Γ_k or Π_k.
- In Π_k: delete universal vertex to obtain a Γ_k.
- We say: a fork is a tree with exactly three leaves, a semi-fork is a graph obtained from a K_3 by appending disjoint paths of length at least 1 at each vertex of K_3.
- Using a constant size part of Γ_k we find an induced fork or semi-fork in G as shown below.



Proof sketch, 3

- In the (huge) Γ_k we combine the forks and semi-forks into a stone wall – an ‘untidy mix’ of a wall and the line graph of a wall:

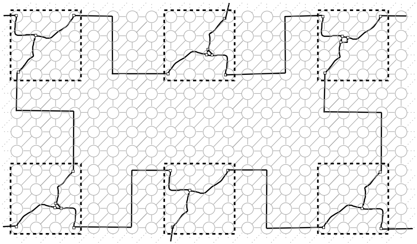


Figure: Just another brick in the wall...

Proof sketch, 4

We show:

Lemma (tidying up)

For every integer $r \geq 2$ there exists an integer $n = n(r)$ such that every $(n \times n)$-stone wall contains an $(r \times r)$-wall or an $(r \times r)$-co-wall as induced subgraph.

The proof uses a variation of Ramsey's theorem for bipartite graphs:

Theorem (Beineke and Schwenk 1975)

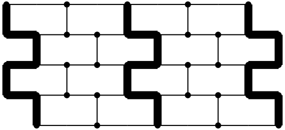
For every integer $r \geq 1$ there exists an integer $n = n(r)$, such that any 2-edge-coloring of the complete bipartite graph $K_{n,n}$ contains a monochromatic $K_{r,r}$.

Proof sketch of the tidying-up lemma, 1

Lemma (tidying up)

For every integer $r \geq 2$ there exists an integer $n = n(r)$ such that every $(n \times n)$-stone wall contains an $(r \times r)$-wall or an $(r \times r)$-co-wall as induced subgraph.

- Given an $(n \times n)$-stone wall W, define a wall W' by contracting each triangle of W into a vertex, color that vertex 'red'. All other degree-3-vertices of W' are colored 'green'.
- Define a complete bipartite graph H with $V(H) = A \cup B$, where $A := \{$ horizontal paths of $W'\}$, $B := \{$ vertical paths in $W'\}$.
- Note: each vertical path has two colored vertices in common with each horizontal path.



Proof sketch of the tidying-up lemma, 2

- Color the edges of H with three colors. Let $P \in A$ be a horizontal path and let $Q \in B$ be a vertical path.

  1. If $V(P) \cap V(Q)$ contains two green vertices, color PQ green.
  2. If $V(P) \cap V(Q)$ contains two red vertices, color PQ red.
  3. If $V(P) \cap V(Q)$ contains green and red, color PQ black.

- [Beineke & Schwenk]: H contains a large monochromatic complete bipartite subgraph H'

  Case 1: We obtain a large wall
  Case 2: We obtain a large co-wall.
  Case 3: Use local rerouting to obtain a large wall.

□

'Induced wall theorem' for graphs excluding a minor

Theorem (P. Aboulker, I. A., E. Kim, N. L. D. Sintiari, N. Trotignon, 2020)

Given H, ex. a function f such that for every H-minor-free G and k:

- *Tree-width$(G) \leq f(k)$, or*
- *G contains a $(k \times k)$-wall or a $(k \times k)$-co-wall as induced subgraph.*

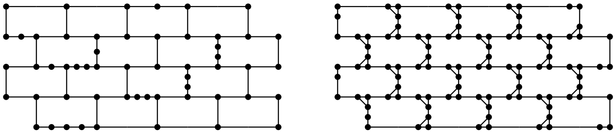


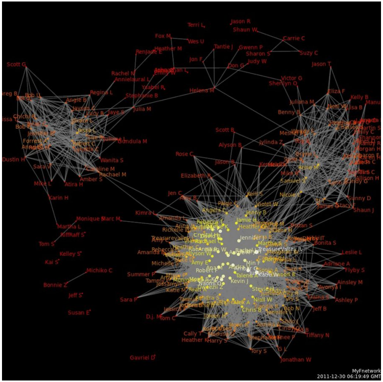
Figure: A (5×5)-wall and a (5×5)-co-wall.

Contents

Motivation

'Efficiency' when the data set is huge:

Even reading the whole input just once can be too expensive.



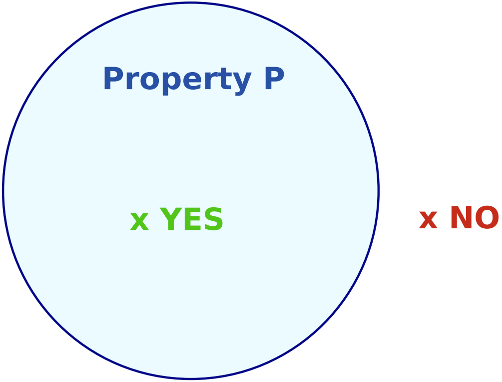
Data visualization of Facebook relationships

Theorem (B. Courcelle 1990)

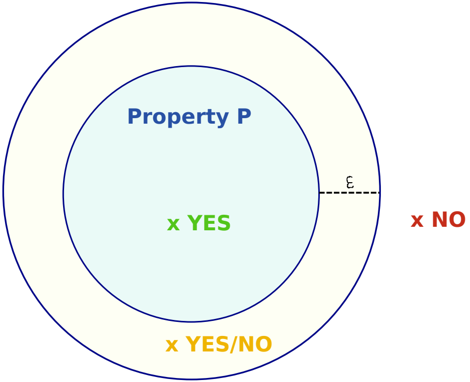
Let $t \in \mathbb{N}$, and $\mathcal{C}$ be a class of graphs of tree-width $\leq t$. Every property expressible in CMSO *is decidable in linear running time on $\mathcal{C}$.*

Can we be faster (sacrificing some accuracy)?

Decision Problems



Property Testing = Relaxation of Decision Problems



On inputs that have the property: YES with probability at least $2/3$.

On ε-far inputs: NO with probability at least $2/3$.

Aim: extremely efficient.

Bounded-degree model

By [O. Goldreich and D. Ron. Property Testing in Bounded Degree Graphs, 2002]

All graphs have degree $\leq d$.

- Let $\varepsilon \in [0, 1]$.
  Graphs G and H, both on n vertices, are ε-close, if we can make them isomorphic by modifying up to εdn edges of G or H.
  Edge modification = insertion/deletion
- If G, H are not ε-*close*, then they are ε-far.
- A graph G is ε-close to a class $\mathcal{C}$ if G is ε-close to some $H \in \mathcal{C}$.
  Otherwise, G is ε-far from $\mathcal{C}$.

Algorithms with oracle access

- Input: the number n of vertices of G, and
- Oracle access to G
  - *Query:* v, for $v \in V(G)$
  - *Answer:* the 1-neighbourhood of vertex v
- The running time $=$ running time w.r.t. n.
- The query complexity $=$ number of oracle queries w.r.t. n.

Examples

Theorem (Goldreich, Ron, 2002)

On bounded degree graphs:

Testable with constant query complexity and running time:

- *k-edge-connectivity*
- *being Eulerian*
- *subgraph-freeness*
- *induced subgraph-freeness*

Not testable with constant query complexity:

- *Bipartiteness*
- *Expander graphs*

Property testing on bounded tree-width

Theorem (I. A., F. Harwath, 2018)
Let $\mathcal{C}_d^t$ be the class of all t-bounded tree-width graphs of degree $\leq d$.
Every CMSO*-definable property $\mathcal{P} \subseteq \mathcal{C}_d^t$ is uniformly testable with constant query complexity and polylogarithmic running time.*

- Even-hole-freenes can be expressed in *CMSO*.
- *k*-bounded tree-width is testable.

Corollary (T. Abrishami, M. Chudnovsky, K. Vušković, 2020)
Even-hole-freeness is testable in the bounded-degree model of property testing with constant query complexity and polylogarithmic running time.

Contents

Outlook

- Ehf graphs excluding a minor have bounded tree-width.
  Via "induced grid theorem" for minor-free graphs
- Subcubic EHF graphs have bounded tree-width.
  Via decomposition theorem
- (Even-hole, pyramid)-free graphs of degree ≤ 4 have bounded tree-width.
  combining structural properties to show they cannot contain a K_6-minor.
- Implications in Property Testing

Conjecture (P. Aboulker, I. A., E. Kim, N. L. D. Sintiari, N. Trotignon, 2020)

For every $d \in \mathbb{N}$ there is a function $f_d : \mathbb{N} \to \mathbb{N}$ such that every graph with degree at most d and tree-width at least $f_d(k)$ contains a $(k \times k)$-wall or the line graph of a $(k \times k)$-wall as an induced subgraph.

Thank you!

Appendix

Theorem

Let G be a (theta, prism)-free subcubic graph. Then one of the following holds:

- *G is a basic graph;*
- *G has a clique separator of size at most 2;*
- *G has a proper separator.*

Basic graphs: chordless cycle, clique of size at most 4, the cube, a proper wheel, a pyramid, or an extended prism.

Proper separation

A proper separation in a graph G is a triple $(\{a, b\}, X, Y)$ s. t.:

1. $\{a, b\}$, X, Y are disjoint, non-empty and $V(G) = \{a, b\} \cup X \cup Y$.
2. There are no edges from X to Y.
3. a and b are non-adjacent.
4. a and b have exactly two neighbors in X.
5. a and b have exactly one neighbor in Y.
6. There exists a path from a to b with interior in X, and there exists a path from a to b with interior in Y.
7. $G[Y \cup \{a, b\}]$ is not a chordless path from a to b.

Extended prism

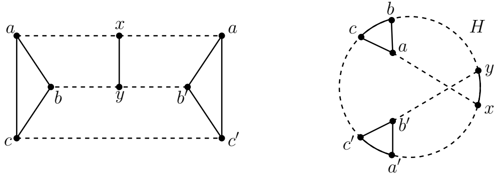


Figure: Two different drawings of an extended prism