Mapping Hypersets into Numbers

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Outline

Sets and Hypersets

Ackermann's bijection

Extending Ackermann's order to Hypersets

(Well-founded) Sets

- We assume the axioms of Zermelo-Fraenkel, including
  - **Axiom of Foundation**: *there are no membership cycles or infinite descending membership chains*
  - **Axiom of Extensionality**: *two sets are equal iff they have the same elements*

- The standard model: von Neumann's cumulative hierarchy of sets, $\mathcal{V}$:
  - $\mathcal{V}_0 = \emptyset$
  - $\mathcal{V}_i = \bigcup_{j<i} \mathscr{P}(\mathcal{V}_j)$
  - $\mathcal{V} = \bigcup_i \mathcal{V}_i$, over all ordinals i

- For example,
  - $\mathcal{V}_1 = \{\emptyset\}$
  - $\mathcal{V}_2 = \{\emptyset, \{\emptyset\}\}$
  - $\mathcal{V}_3 = \Big\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\Big\}, \ldots$

REPRESENTING SETS BY DIRECTED GRAPHS

$$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$$

Representing sets by directed graphs

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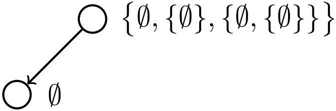
Representing sets by directed graphs



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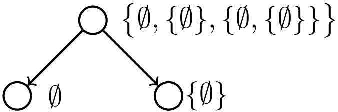

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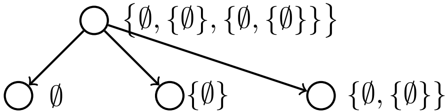

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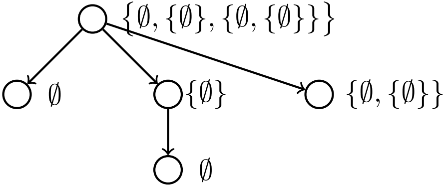

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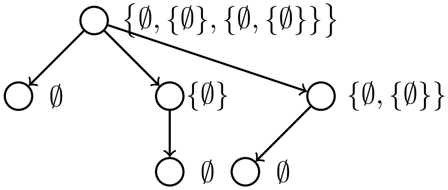

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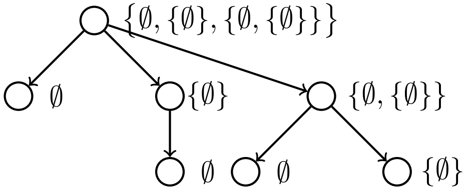

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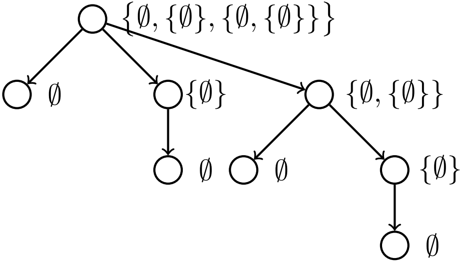

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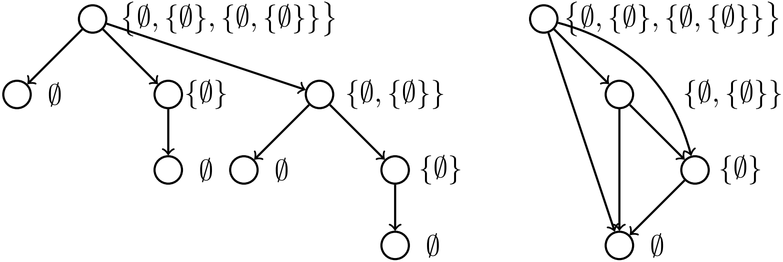

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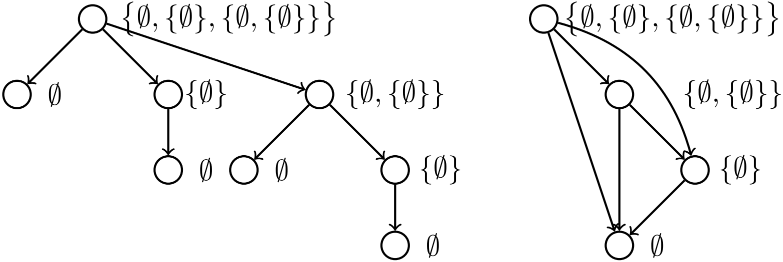

- A node x is *redundant* in an acyclic digraph G if there is another node of G with the same set of out-neighbors as x.

DEFINITION

A set $=_{def}$ a pointed acyclic digraph without *redundant* nodes.

$$x \in y \Longleftrightarrow x \leftarrow y$$

HYPERSETS

Modern modeling approaches may require $\in$ to be cyclic.

DEFINITION

A hyperset $=_{def}$ a pointed digraph without *redundant* nodes.

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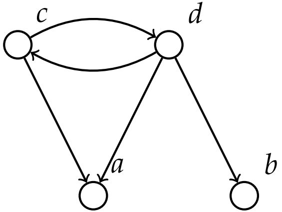
DEFINITION

Given a digraph G, a bisimulation on G is a relation $\flat \subseteq V(G) \times V(G)$ iff for all $x, y \in V(G)$ s.t. $x \flat y$

- $\forall x' \ (x \to x') \Rightarrow \exists y' \ (y \to y' \ \wedge \ x' \flat y')$;
- $\forall y' \ (y \to y') \Rightarrow \exists x' \ (x \to x' \ \wedge \ x' \flat y')$.

- Bisimilarity in G is an equivalence relation on G.

BISIMULATION - EXAMPLE



The bisimilarity relation is **[**a, b**]** **[**c, d**]**

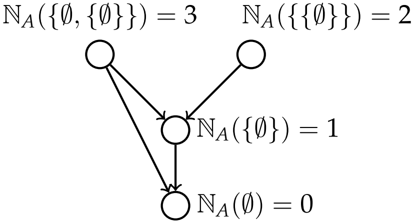
Ackermann's encoding

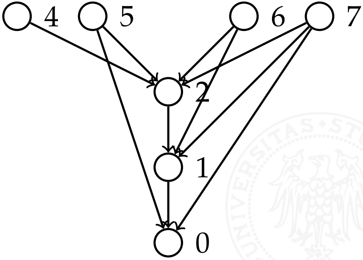
Let

- HF be the set of all hereditarily finite sets.
- $\overline{\mathsf{HF}} \supsetneq \mathsf{HF}$ be the set of all hereditarily finite hypersets.

Ackermann's bijection (1937) $\mathbb{N}_A : \mathsf{HF} \rightarrow \mathbb{N}$, where

$$\mathbb{N}_A(F) =_{Def} \sum_{h \in F} 2^{\mathbb{N}_A(h)}.$$





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It induces the linear order $\prec_A$ on HF, also expressible as

$$F \prec_A F' \Leftrightarrow_{Def} \mathbb{N}_A(F) < \mathbb{N}_A(F') \Leftrightarrow \max_{\prec_A}(F \setminus F') \prec_A \max_{\prec_A}(F' \setminus F).$$

MAIN CONTRIBUTION

A *natural* extension of $\mathbb{N}_A$ to $\overline{\mathsf{HF}}$.

APPLICATIONS OF $\mathbb{N}_A$

RECURSION, DECIDABILITY RESULTS

Sets: Since $\prec_A$ is a well-founded order, it is easy to do recursion over HF.
Hypersets: Ad-hoc solutions.

COUNTING (COMBINATORIAL ENUMERATION)

Sets: We know the number of transitive sets with n elements, because of an Ackermann-like bijection between them and the set $\{(x_0, x_1, \ldots, x_{n-1}) \ : \ x_0 = 0, x_{i-1} < x_i < 2^i, 1 \leq i \leq n-1\}$.
(Peddicord, 1962)
Hypersets: **OPEN** (to our knowledge)

APPLICATIONS OF $\mathbb{N}_A$ (2)

ALGORITHMICS

Sets: The problem "Given an acyclic digraph G on n nodes and m arcs, compute the maximum bisimulation on G" can be solved in time $\mathcal{O}(m)$ by trying to find $\prec_A$ on G.
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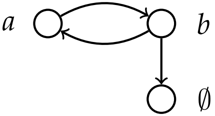
Linear algorithm?

- when $E(G)$ corresponds to a function: $\mathcal{O}(m)$ (Paige, Tajan, Bonic, 1985);
- the general case: **OPEN**

A naive extension of $\prec_A$ to $\overline{\mathsf{HF}}$ fails



Consider $a = \{b\}, b = \{a, \emptyset\}$.

Note that $\emptyset \prec a$.

$a \prec b \Leftrightarrow \max_{\prec}\{b\} \prec \max_{\prec}\{a, \emptyset\} \Leftrightarrow b \prec a$.

Our approach

- Instead of defining first $\mathbb{Q}_A$, and then taking the induced order, we do the opposite.

OUR APPROACH

- Instead of defining first $\mathbb{Q}_A$, and then taking the induced order, we do the opposite.
- How do we do it?
  $\prec_A$ over G acyclic $\Longrightarrow$ algorithm to compute the maximum bisimulation over G (DPP'04).
  $\prec_H$ over any G $\Longleftarrow$ algorithm to compute the maximum bisimulation over G (PT'87)

OUR APPROACH

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  $\prec_A$ over G acyclic $\Longrightarrow$ algorithm to compute the maximum bisimulation over G (DPP'04).
  $\prec_H$ over any G $\Longleftarrow$ algorithm to compute the maximum bisimulation over G (PT'87)
- Finally, for all $a \in \overline{\mathsf{HF}}$, define:

$$\mathbb{Z}_A(a) \quad = \quad \begin{cases} |\{b \in \mathsf{HF} \ : \ b \prec_H a\}| = \mathbb{N}_A(a) & \text{if } a \in \mathsf{HF}, \\ -|\{b \in \overline{\mathsf{HF}} \setminus \mathsf{HF} \ : \ b \prec_H a\}| - 1 & \text{if } a \in \overline{\mathsf{HF}} \setminus \mathsf{HF}. \end{cases}$$

$$\mathbb{Q}_A : \overline{\mathsf{HF}} \to \left\{ \frac{m}{2^n} \ : \ n, m \in \mathbb{N} \right\}, \quad \mathbb{Q}_A(a) = \sum_{b \in a} 2^{\mathbb{Z}_A(b)}.$$

STABLE PARTITIONING

DEFINITION

- Given a set V, a relation $E \subseteq V \times V$, and a partition P of V, P is stable w.r.t. E iff $\forall B_1, B_2 \in P\big(B_1 \subseteq E^{-1}(B_2) \vee B_1 \cap E^{-1}(B_2) = \emptyset)\big)$
- Given a set V and partitions P, Q of V,
  - P refines Q iff $\forall B \in P\ \exists C \in Q\big(B \subseteq C\big)$
  - Q is coarser than P iff P refines Q

- Paige-Tarjan's 1987 algorithm solves the problem of finding the coarsest partition of $V(G)$, stable w.r.t. $E(G)$
- Finding the maximum bisimulation is equivalent to the coarsest partition problem (Kannellakis, Smolka, 1990)

The splitting technique

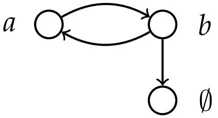
The basic primitive of the PT'87 algorithm

Given $S \in P$, replace P by

$$\{B \cap E^{-1}(S), B \setminus E^{-1}(S) \ : \ B \in P\}$$

We will do the opposite: Given $T \in P$, replace T by all the equivalence classes of T induced by

$$x \sim_E y \Leftrightarrow_{Def} \forall B \in P\big(x \in E^{-1}(B) \leftrightarrow y \in E^{-1}(B)\big)$$



$[\emptyset, a, b]$

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THE RANK NOTION FOR HF

RANK

Define $\mathsf{rk} : \mathsf{HF} \to \mathbb{N}$ as

$$\mathsf{rk}(v) = \max\{1 + \mathsf{rk}(u) \ : \ \forall u \in v\}.$$

It holds that $\mathsf{rk}(u) < \mathsf{rk}(v) \Rightarrow u \prec_A v$.

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We can obtain the partition of HF whose blocks are the rank-equality classes: Start with $P_0 = \{\mathsf{HF}\}$

- That there is exactly one infinite block S_n of P_n
- S_n is a culprit of the instability of P_n
- $P_{n+1} = (P_n \setminus \{S_n\}) \cup \{\{x \in S_n | x \cap S_n = \emptyset\}, \{x \in S_n | x \cap S_n \neq \emptyset\}\}$

$$[\emptyset, \quad \{\emptyset\}, \quad \{\{\emptyset\}\}, \quad \{\emptyset, \{\emptyset\}\}, \quad \{\{\{\emptyset\}\}\}, \ldots]$$

$$[\emptyset] \prec_A [\{\emptyset\}, \quad \{\{\emptyset\}\}, \quad \{\emptyset, \{\emptyset\}\}, \quad \{\{\{\emptyset\}\}\}, \ldots]$$

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A NEW DEFINITION OF $\prec_A$

- Define a countable sequence $(\mathcal{X}^n)_{n\in\mathbb{N}}$ of ordered partitions, $\mathcal{X}^n = \{X_i^n \,:\, i \in \mathbb{N}\}$
- Each $\mathcal{X}^{n+1}$ is an ordered refinement of $\mathcal{X}^n$
- Initially, $\mathcal{X}_i^0 = \{x \in \mathsf{HF} \,:\, \mathsf{rk}(x) = i\}$

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- Consider the smallest index h such that $\mathcal{X}_h^n$ can be split, and the relation on $\mathcal{X}_h^n$

$$x \sim_{\ni} y \Leftrightarrow_{Def} \forall k \big(X_k^n \cap x \neq \emptyset \leftrightarrow X_k^n \cap y \neq \emptyset\big)$$

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- Given two $\sim_{\ni}$-classes $Z', Z \subseteq X_h^n$, put Z' *before* Z iff, for $x \in Z'$ and $y \in Z$, the largest mismatch position k between x, y 'favors' y, i.e.

$$X_k^n \cap x = \emptyset \;\wedge\; X_k^n \cap y \neq \emptyset \;\wedge\; \forall j > k(\, X_j^n \cap x = \emptyset \leftrightarrow X_j^n \cap y = \emptyset\,)$$

A NEW DEFINITION OF $\prec_A$ (2)

THEOREM

At limit,

- *Every* $x \in \mathsf{HF}$ *remains in a singleton block.*
- *Ackermann's order is the limit of the* $\mathcal{X}^n$*'s.*

Why is the Rank important?

- It guarantees *correctness*, i.e., the order is *à la* Ackermann.
- It guarantees *convergence*, i.e., the blocks become singletons after ω steps.
  - The set $\{x \in \mathsf{HF} \ : \ \mathsf{rk}(x) = i\}$ is finite, $\forall i$.

A RANK NOTION FOR $\overline{\mathsf{HF}}$

DEFINITION

Define $\mathsf{rk} : \overline{\mathsf{HF}} \to \mathbb{N}$ as

$$\mathsf{rk}(x) \quad = \quad \text{the maximum length of all simple directed paths in } x\text{, issuing from the point of } x.$$

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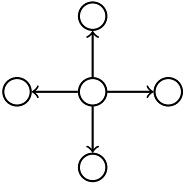
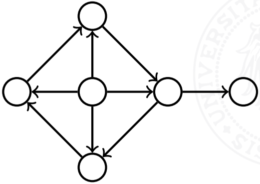
LEMMA

There exists a function $r(n)$ s.t. $|\{x \in \overline{\mathsf{HF}} \mid \mathsf{rk}(x) \leq n\}| < r(n)$.

For arbitrary graphs of bounded 'diameter' this is not the case:





THE ORDER FOR HYPERSETS

Start with the partition $\mathcal{X}^0 = \{X_i^0 : i \in \mathbb{N}\}$, where $X_i^0 = \{x \in \overline{\mathsf{HF}} : \mathsf{rk}(x) = i\}$, and iteratively apply the operation of *ordered refinement*.

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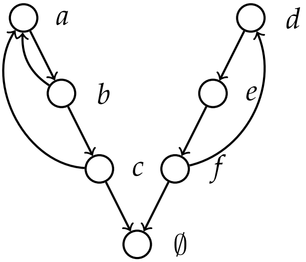

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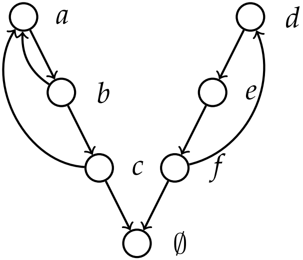

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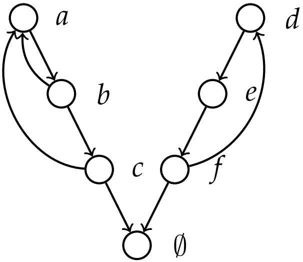

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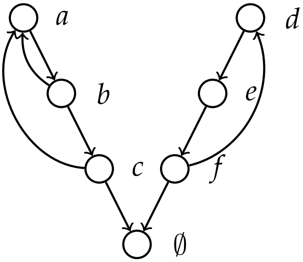

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Conclusions

- A step closer in better understanding bisimulations, hypersets
  - Connection between recognizing a hyperset, counting, and enumerating them
- A new definition of Ackermann's order for HF
- A new notion of *rank* for hypersets
  - Used here to get correctness and convergence
  - Any other *adequate rank* notion may give another order on $\overline{\mathsf{HF}} \setminus \mathsf{HF}$
- Possible applications
  - To show that any hyperset can be transformed into any other one by adding/removing arcs (useful in random generation by Markov Chains)
- Deeper exploration of the connection between sets and numbers
  - Which sets have a prime number encoding?