Towards a combinatorial characterization of equistable graphs

(Partial results on a conjecture of Orlin)

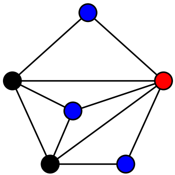
Martin Milanič

UP FAMNIT and UP PINT, University of Primorska

Raziskovalni matematični seminar, FAMNIT, 28. marec 2011

Graphs and stable sets

- $G = (V, E)$ - a finite simple undirected graph
- stable (independent) set: a subset $S \subseteq V$ of pairwise non-adjacent vertices
- a stable set is maximal if it is not contained in any larger stable set



Equistable graphs

Definition

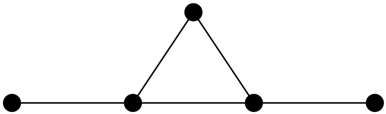
A graph $G = (V, E)$ is **equistable** if there exists a function $w : V \to \mathbb{N}$ and a positive integer t such that

$\forall S \subseteq V$:

S is a maximal stable set in G $\Leftrightarrow$ $w(S) = \sum_{v \in S} w(v) = t$.

Equistable graphs: example

The following graph is equistable:



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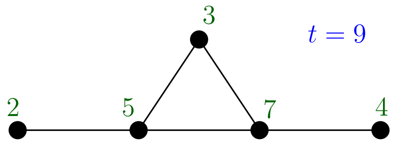

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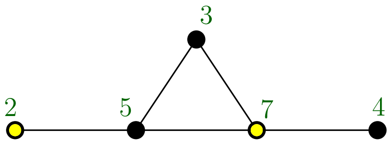

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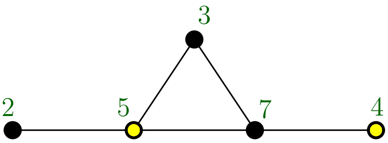

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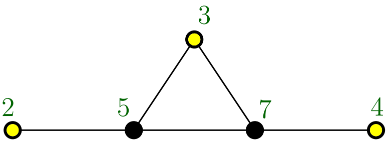
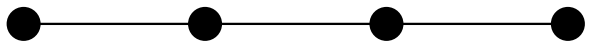


Equistable graphs: example

The following graph is not equistable:



Equistable graphs: example

The following graph is not equistable:

If

$$\begin{aligned} w_1 + w_3 &= t \\ w_2 + w_4 &= t \\ w_1 + w_4 &= t \end{aligned}$$

then

$$w_2 + w_3 = t\,.$$

Equistable graphs: example

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Equistable graphs: motivation

- **threshold graphs** (Chvátal-Hammer 1977):
  $\exists w, t$ s.t. $S \subseteq V$ stable $\Leftrightarrow w(S) \leq t$
- **equistable graphs** (Payan 1980):
  $\exists w, t$ s.t. $S \subseteq V$ maximal stable $\Leftrightarrow w(S) = t$

Equistable graphs generalize:

- threshold graphs (Payan, 1980);
- co-graphs (graphs without an induced 3-edge path) (Mahadev-Peled-Sun, 1994).

Equistable graphs: motivation

- **threshold graphs** (Chvátal-Hammer 1977):
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Equistable graphs generalize:

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- co-graphs (graphs without an induced 3-edge path) (Mahadev-Peled-Sun, 1994).

General partition graphs

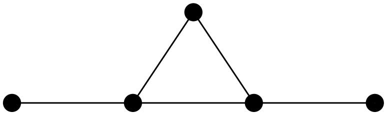
Definition

A graph $G = (V, E)$ is a **general partition graph (gpg)** if there exists a finite set U and an assignment of nonempty subsets $U_x \subseteq U$ to vertices of V such that

- $xy \in E$ if and only if $U_x \cap U_y \neq \emptyset$, and
- for every maximal stable set S in G, the set $\{U_x : x \in S\}$ forms a partition of U.

General partition graphs: example

The following graph is a gpg:



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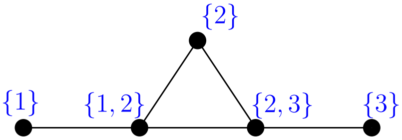


$U = \{1, 2, 3\}$

General partition graphs: a characterization

Theorem (McAvaney-Robertson-DeTemple, 1993)

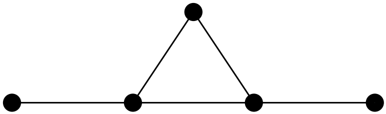
For every graph G, the following are equivalent:

- *G is a gpg,*
- *every edge of G is contained in a strong clique.*

strong clique = a clique meeting all maximal stable sets

General partition graphs: example

The following graph is a gpg (every edge is contained in a strong clique):



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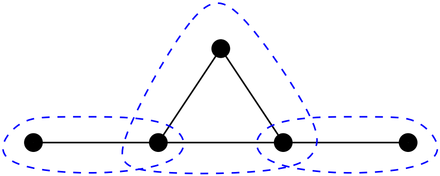
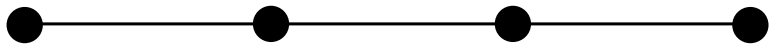


General partition graphs: example

The following graph is not a gpg (there exists an edge not contained in any strong clique):



General partition graphs: example

The following graph is not a gpg (there exists an edge not contained in any strong clique):

Triangle graphs

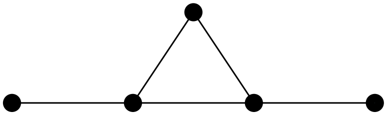
Definition

A graph $G = (V, E)$ is a **triangle graph** if it satisfies the following triangle condition:

- for every maximal stable set S in G and every edge $uv \in E(G - S)$, u and v have a common neighbor in S.

Triangle graphs: example

The following graph is a triangle graph:



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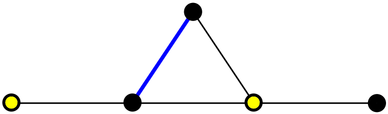

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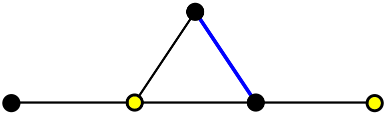

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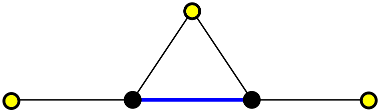

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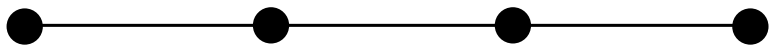

Triangle graphs: example

The following graph is not a triangle graph:

Inclusion relations among these classes

The following inclusion relations hold:

$$\begin{array}{c}\text{general partition graphs}\\ \subseteq\\ \text{strongly equistable graphs}\\ \subseteq\\ \text{equistable graphs}\\ \subset\\ \text{triangle graphs}\end{array}$$

A condition equivalent to the triangle condition

In the equistable graphs literature, the triangle condition was replaced with the following equivalent condition:

"absence of a **bad** P_4":

For each induced P_4 on the vertices a, b, c, d, each maximal stable set containing the end-vertices a and d has a common neighbor of the middle vertices b and c.

Inclusion relations among these classes

The following inclusion relations hold:

$$
\begin{array}{c}
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General partition graphs are equistable

Theorem (Jim Orlin, 2009)

Every gpg is equistable.

Proof idea:

Let $(U_x : x \in V)$ with $U = \{u_1, \ldots, u_k\}$ be a set system realizing G.

Let $n = |V(G)|$ and assign to each $x \in V$ weight

$$w(x) = \sum\{n^j : u_j \in U_x\},$$

also let

$$t = \sum_{j=1}^{k} n^j.$$

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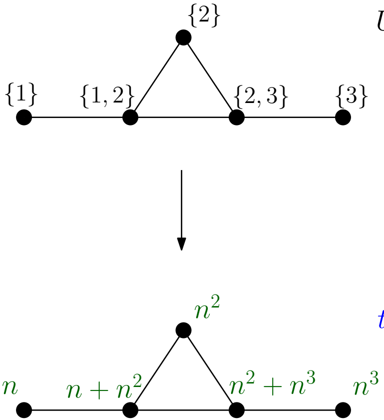
Example



$U = \{1, 2, 3\}$

$t = n + n^2 + n^3$

General partition graphs are strongly equistable

Theorem (Jim Orlin, 2009)

Every gpg is equistable.

Theorem (McAvaney-Robertson-DeTemple, 1993)

G is a gpg if and only if every edge of G is contained in a strong clique.

Theorem (Mahadev-Peled-Sun, 1994)

Every equistable with a strong clique is strongly equistable.

Corollary

Every gpg is strongly equistable.

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Inclusion relations among these classes

$$\begin{array}{c}\text{general partition graphs}\\ \subseteq \\ \text{strongly equistable graphs}\\ \subseteq \\ \text{equistable graphs}\\ \subset \\ \text{triangle graphs}\end{array}$$

Conjecture (Mahadev-Peled-Sun, 1994)

Every equistable graph is strongly equistable.

Conjecture (Jim Orlin, 2009)

Every equistable graph is a gpg.

If true, Orlin's conjecture would provide a combinatorial characterization of equistable graphs.

Known results

Orlin's conjecture holds within the following graph classes:

- chordal graphs,
- graphs obtained from triangle-free graphs by gluing chordal graphs along edges,
- outerplanar graphs,
- series-parallel graphs $\equiv K_4$-minor-free graphs.

Known results

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Inclusion relations among these classes

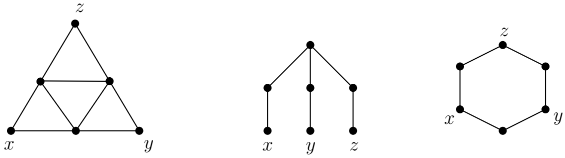
general partition graphs	combinatorial definition
$\subseteq$	
strongly equistable graphs	algebraic definition
$\subseteq$	
equistable graphs	algebraic definition
$\subset$	
triangle graphs	combinatorial definition

Our goal:

Identify further combinatorially defined graph classes $\mathcal{C}$ such that within $\mathcal{C}$, some of the above inclusions become equalities.

First example: AT-free graphs

Asteroidal triple (AT): a triple of vertices x, y, z such that for every pair of them there is a path connecting the two vertices that avoids the neighborhood of the remaining vertex.



Theorem (Kloks-Lee-Liu-Müller, 2003)

Every AT-free triangle graph is a gpg.

Corollary

Orlin's conjecture holds within the class of AT-free graphs.

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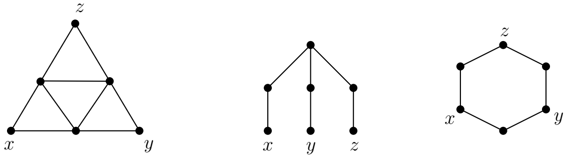

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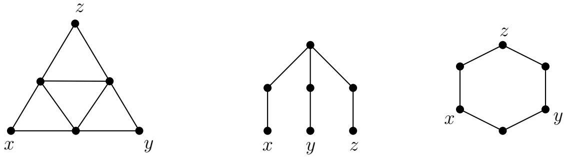

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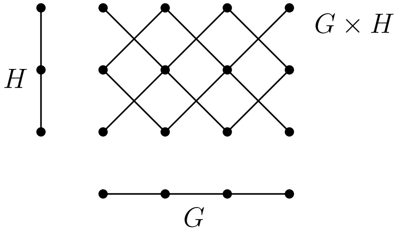
Tensor products

Definition

The **tensor** (or **direct**) **product** of graphs G and H is the graph $G \times H$ such that:

- $V(G \times H) = V(G) \times V(H)$,
- $(u,x)(v,y) \in E(G \times H)$ if and only if $uv \in E(G) \wedge xy \in E(H)$.



Equistable tensor products

Theorem

Let $G = G_1 \times G_2$, where G_1, G_2 are connected, with more than just one vertex. The following are equivalent:

(i) *G is a gpg.*

(ii) *G is strongly equistable.*

(iii) *G is equistable.*

(iv) *$\exists m \geq 2$ such that G_1 and G_2 are complete m-partite graphs.*

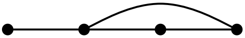
Proof sketch of $(iii) \Rightarrow (iv)$

$G = G_1 \times G_2$ equistable.

Claim

No induced subgraph of G_1 is isomorphic to P_4 or a paw.

P_4



paw

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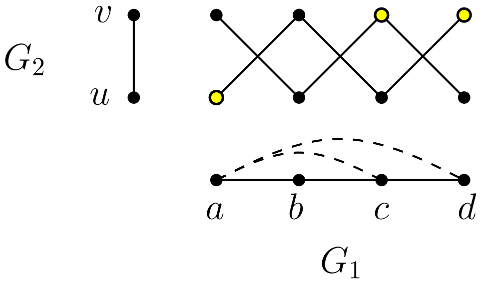
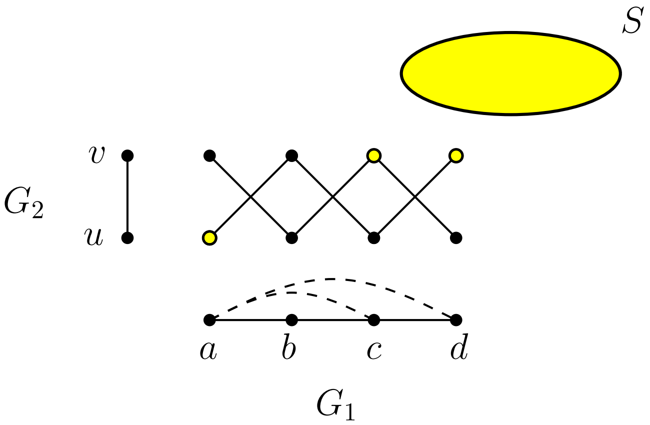


Proof sketch of $(iii) \Rightarrow (iv)$

S: a maximal stable set of G containing (a, u), (c, v), (d, v)



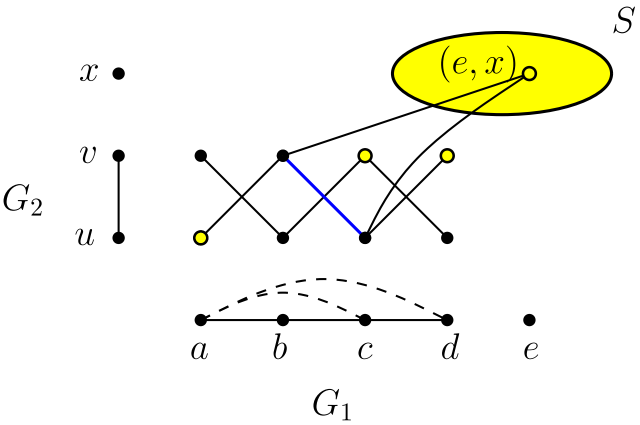
Proof sketch of $(iii) \Rightarrow (iv)$

Every equistable graph satisfies the triangle condition:

- for every maximal stable set S in G and every edge $uv \in E(G - S)$, u and v have a common neighbor in S.

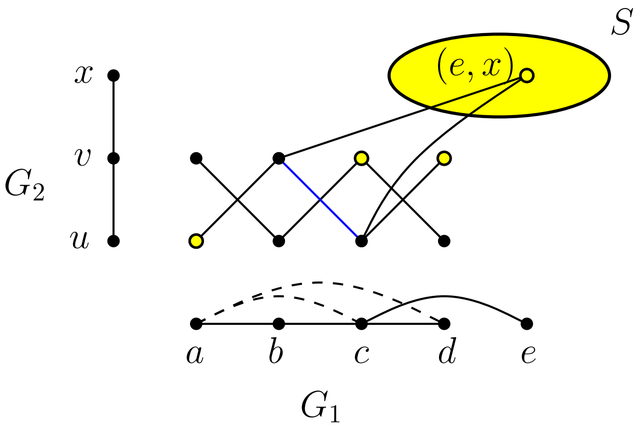
Proof sketch of $(iii) \Rightarrow (iv)$

$(e, x) \in S$: a common neighbor of (b, v) and (c, u)



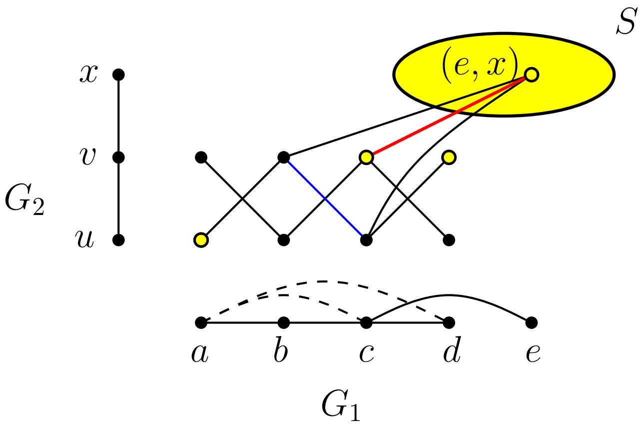
Proof sketch of $(iii) \Rightarrow (iv)$

$(e,x)(c,u) \in E(G_1 \times G_2) \Rightarrow ce \in E(G_1)$
$(e,x)(b,v) \in E(G_1 \times G_2) \Rightarrow vx \in E(G_2)$



Proof sketch of $(iii) \Rightarrow (iv)$

$$ce \in E(G_1) \wedge vx \in E(G_2) \Rightarrow (c,v)(e,x) \in E(G_1 \times G_2)$$



Contradiction with the fact that S is a stable set.

Proof sketch of $(iii) \Rightarrow (iv)$

$G = G_1 \times G_2$ equistable.

Claim

G_1 is (P_4, paw)-free.
G_2 is (P_4, paw)-free.

Lemma

Every connected (P_4, paw)-free graph is complete multipartite.

So both G_1 and G_2 are complete multipartite.

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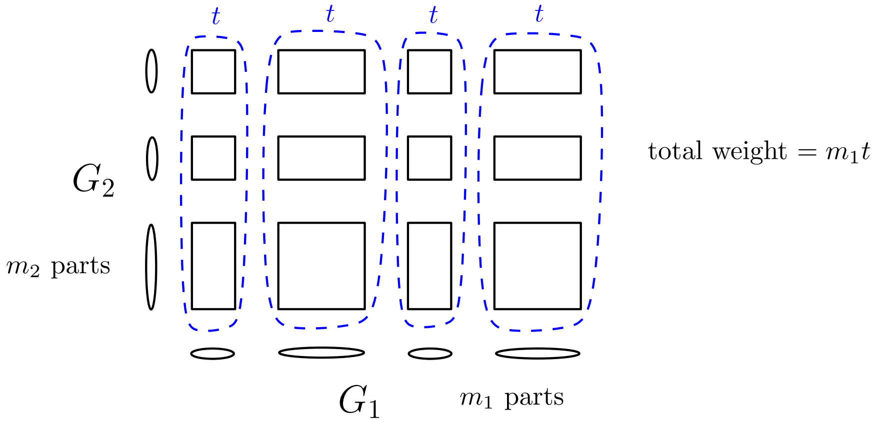
G_1 and G_2 have the same number of parts:



Proof sketch of (*iii*) $\Rightarrow$ (*iv*)

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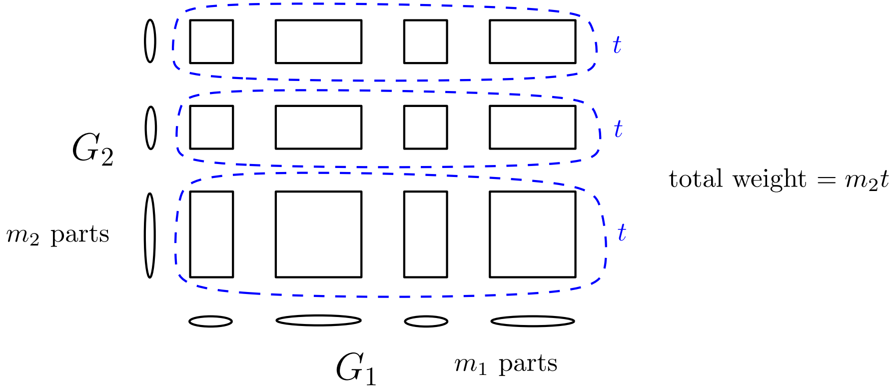

Cartesian products

Definition

The **Cartesian product** of graphs G and H is the graph $G\Box H$ such that:

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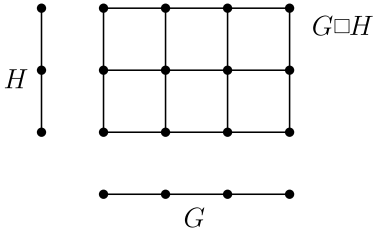
- $(u,x)(v,y) \in E(G\Box H)$ if and only if $(u = v \wedge xy \in E(H)) \vee (x = y \wedge uv \in E(G))$.



Equistable Cartesian products

Theorem

Let $G = G_1 \square G_2$, where G_1, G_2 are connected, with more than just one vertex. The following are equivalent:

(i) *G is a gpg.*

(ii) *G is strongly equistable.*

(iii) *G is equistable.*

(iv) *G is a triangle graph.*

(v) *$\exists m \geq 2$ such that $G_1 \cong G_2 \cong K_m$.*

Proof sketch of $(iv) \Rightarrow (v)$

For a graph K, we denote:

- by $\delta(K)$ the minimum vertex degree,
- by $\Delta(K)$ the maximum vertex degree,
- for $uv \in E(K)$, let $\lambda(uv) = |N(u) \cap N(v)|$,
- $\lambda(K) = \min\{\lambda(uv) \,:\, uv \in E(K)\}$.

Observation

$$\lambda(K) \leq \delta(K) - 1\,.$$

Proof sketch of $(iv) \Rightarrow (v)$

$G = G_1 \Box G_2$ triangle graph

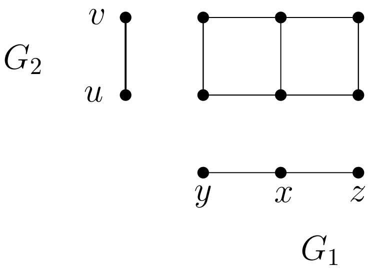
Claim

$\lambda(G_2) \geq \Delta(G_1) - 1.$

Suppose for a contradiction that $\lambda(G_2) \leq \Delta(G_1) - 2.$

Proof sketch of $(iv) \Rightarrow (v)$

$\lambda(G_2) \leq \Delta(G_1) - 2$.
x: vertex of maximum degree in G_1 ($d(x) \geq 2$)
y, z: two neighbors of x,
$uv \in E(G_2)$: $\lambda(uv) = \lambda(G_2)$.



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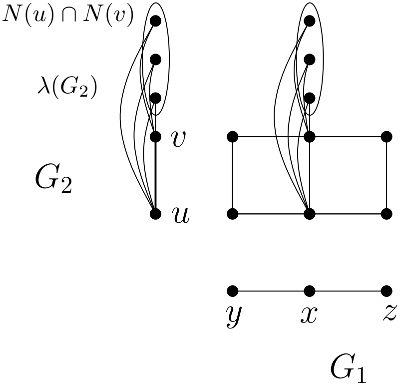

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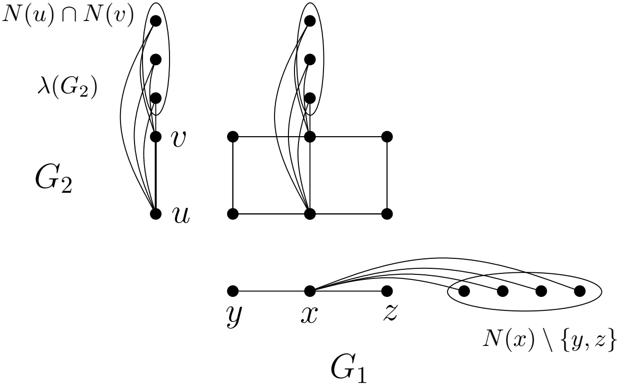
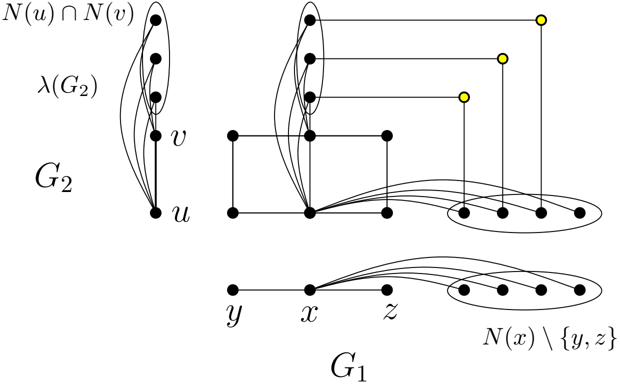


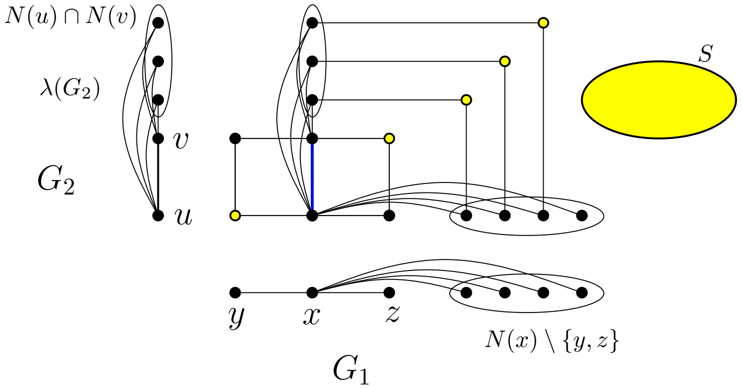
Proof sketch of $(iv) \Rightarrow (v)$

$\lambda(G_2) \leq \Delta(G_1) - 2.$
$|N(x) \setminus \{y, z\}| \geq \lambda(G_2).$



Proof sketch of $(iv) \Rightarrow (v)$

$\lambda(G_2) \leq \Delta(G_1) - 2$.
S: a maximal stable set containing all yellow vertices.



Contradiction.

Proof sketch of $(iv) \Rightarrow (v)$

Claim

$\lambda(G_2) \geq \Delta(G_1) - 1.$

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Consequently

$$\lambda(G_1) \geq \Delta(G_2) - 1 \geq \delta(G_2) - 1 \geq \lambda(G_2) \geq \Delta(G_1) - 1 \geq \delta(G_1) - 1 \geq \lambda(G_1)\,.$$

Equalities hold, therefore:

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- G_1 and G_2 are both regular of the same degree $m - 1$.
- $\lambda(G_i) = m - 2$ implies that $G_1 \cong G_2 \cong K_m$.

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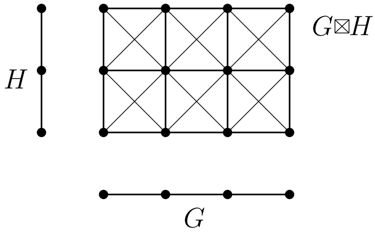
Strong products

Definition

The **strong product** of graphs G and H is the graph $G \boxtimes H$ such that:

- $V(G \boxtimes H) = V(G) \times V(H)$,
- $E(G \boxtimes H) = E(G \times H) \cup E(G \Box H)$.



Equistable strong products

Theorem (McAvaney-Robertson-DeTemple, 1993)

If $G_1 \boxtimes G_2$ is a gpg, then G_1 and G_2 are gpgs.

Proposition

The general partition (equistable, strongly equistable, triangle) graphs are not closed under the strong product.

Counterexample: $(K_3 \square K_3) \boxtimes (K_3 \square K_3)$.

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Equistable strong products

Theorem

If $G_1 \boxtimes G_2$ is a triangle graph, then G_1 and G_2 are triangle graphs.

Theorem

If every edge of G_1 and G_2 is contained in a simplicial clique, then $G_1 \boxtimes G_2$ is a gpg.

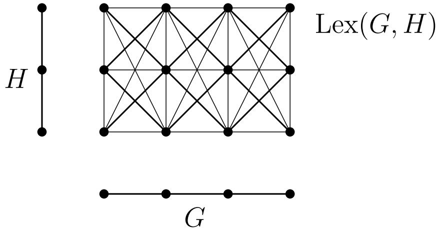
simplicial clique: a clique of the form $K = N(v) \cup \{v\}$.

Lexicographic products

Definition

The **lexicographic product** of graphs G and H is the graph $\mathrm{Lex}(G, H)$ such that:

- $V(\mathrm{Lex}(G, H)) = V(G) \times V(H)$,
- $(u, x)(v, y) \in E(\mathrm{Lex}(G, H))$ if and only if $(uv \in E(G)) \vee (u = v \wedge xy \in E(H))$.



Lexicographic products

Theorem (McAvaney-Robertson-DeTemple, 1993)

$\mathrm{Lex}(G_1, G_2)$ *is a gpg if and only if* G_1 *and* G_2 *are gpgs.*

Theorem

$\mathrm{Lex}(G_1, G_2)$ *is a triangle graph if and only if* G_1 *and* G_2 *are triangle graphs.*

Theorem

(*i*) *If* $\mathrm{Lex}(G_1, G_2)$ *is equistable, then* G_1 *and* G_2 *are equistable.*

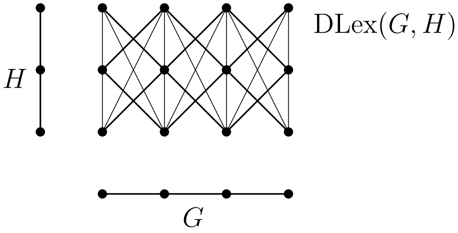
(*ii*) *If* G_1 *and* G_2 *are equistable and* G_2 *contains an isolated vertex, then* $\mathrm{Lex}(G_1, G_2)$ *is equistable.*

Deleted lexicographic products

Definition

The **deleted lexicographic product** of graphs G and H is the graph $\mathrm{DLex}(G,H)$ such that:

- $V(\mathrm{DLex}(G,H)) = V(G) \times V(H)$,
- $(u,x)(v,y) \in E(\mathrm{DLex}(G,H))$ if and only if $(uv \in E(G) \wedge x \neq y) \vee (u = v \wedge xy \in E(H))$.



Deleted lexicographic products

Theorem

Let G be connected, triangle-free, with at least two vertices, H with at least one edge. The following are equivalent:

(i) $\mathrm{DLex}(G, H)$ *is a gpg.*

(ii) $\mathrm{DLex}(G, H)$ *is strongly equistable.*

(iii) $\mathrm{DLex}(G, H)$ *is equistable.*

(iv) $\mathrm{DLex}(G, H)$ *is a triangle graph.*

(v) *Either* $G = K_2$ *and the complement of* H *is of maximum degree at most 1, or* $G \neq K_2$ *is complete bipartite and* $H = K_{t\times 2}$ *for some* $t \geq 2$*.*

Open problems

Determine the complexity of recognizing equistable graphs.

Conjecture (Mahadev-Peled-Sun, 1994)

Every equistable graph is strongly equistable.

Conjecture (Jim Orlin, 2009)

Every equistable graph is a gpg.

Conjecture (ŠM-MM, 2011)

Every equistable graph contains a strong clique.

The end

Thank you