Generalized Cayley graphs

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Overview

- Basic definitions
- Cayley graphs
- Generalized Cayley graphs
- Automorphism of Generalized Cayley graphs
- Non-Cayley vertex-transitive generalized Cayley graphs

Basics about graphs

A graph is an ordered pair $\Gamma = (V, E)$, where V denotes the set of vertices, and E denotes the set of edges of the graph Γ.
Automorphism of a graph Γ is a bijective function $\varphi : V(\Gamma) \to V(\Gamma)$ such that $\{x, y\} \in E(\Gamma) \Leftrightarrow \{\varphi(x), \varphi(y)\} \in E(\Gamma)$. We define the set $Aut(\Gamma)$ to be the set of all automorphisms of the graph Γ.
It is not difficult to see that $Aut(\Gamma)$ is in fact the group with respect to composition of functions, and it is called the automorphism group of Γ.

Group actions

Let G be a group and X a nonempty set. A (left) **group action of G on X** is a binary operator:

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2. $(gh) \circ x = g \circ (h \circ x), \forall g, h \in G, x \in X.$

The group G is said to act on the set X (on the left), and the set X is called a (left) G-set.
Instead of $g \circ x$ we usually just write gx.

Types of actions

Let G act on a set X. For $x \in X$ define a **stabilizer of** x **in** G denoted by G_x as $G_x = \{g \in G | gx = x\}$.

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- **semiregular** $\Leftrightarrow$ for every $x \in X$ the stabilizer G_x is trivial.
- **regular** $\Leftrightarrow$ transitive + semiregular.

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When we speak about graphs we consider the action of the group of the automorphisms on the graph. We say that a graph Γ is

- vertex-transitive $\Leftrightarrow$ $\mathrm{Aut}(\Gamma)$ acts transitively on the vertex set of the graph;
- edge-transitive $\Leftrightarrow$ $\mathrm{Aut}(\Gamma)$ acts transitively on the edge set of the graph;
- arc-transitive $\Leftrightarrow$ $\mathrm{Aut}(\Gamma)$ acts transitively on the arc set of the graph.

Given a group G and a subset S of G such that:

(i) $1 \notin S$,

(ii) $S^{-1} = S$;

the *Cayley graph* $\mathrm{Cay}(G, S)$ of G relative to S has vertex set G and edges of the form $\{g, gs\}$ where $g \in G$ and $s \in S$.

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Example

$G = \mathbb{Z}_6$, $S = \{\pm 1, 3\}$.

If $X = Cay(G,S)$ then the action of G on itself by the left multiplication induces a subgroup of the automorphism group which acts transitively on vertices, hence every Cayley graph is vertex-transitive.

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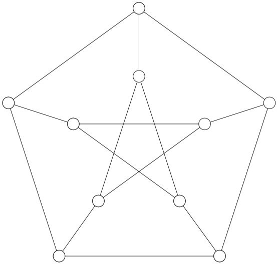
However, not every vertex-transitive graph is Cayley graph. The smallest vertex-transitive graph which is not Cayley is the Petersen graph.



Let G be a finite group, S a non-empty subset of G and α an automorphism of G such that the following conditions are satisfied:

(i) $\alpha^2 = 1$,

(ii) $\alpha(g^{-1})g \notin S$, $(\forall g \in G)$

(iii) $\alpha(S^{-1}) = S$.

Then the *generalized Cayley graph* $X = GC(G, S, \alpha)$ on G with respect to the ordered pair (S, α) is a graph with vertex set G, and edges of form $\{g, \alpha(g)s\}$, where $g \in G$ and $s \in S$.

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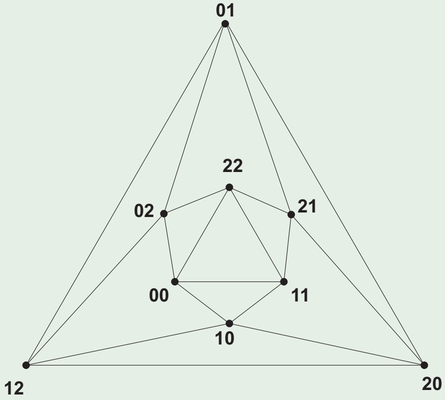
Let $G = \mathbb{Z}_3 \times \mathbb{Z}_3$, $S = \{(1,0),(1,1),(0,2),(2,2)\}$ and $\alpha : (i,j) \mapsto (j,i)$.



History of generalized Cayley graphs

The concept of generalized Cayley graphs was introduced by Marušič, Scapellato and Salvi in 1992. They studied properties of such graphs relative to double coverings of graphs (sometimes called bipartite double cover or canonical double cover).

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Theorem (Marušič, Scapellato, Salvi, 1992)

Let X be a non-bipartite graph. Then its double covering is a Cayley graph if and only if X is a generalized Cayley graph.

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The following problem was also posed.

Problem (Marušič, Scapellato, Salvi, 1992)

Are there generalized Cayley graphs which are not Cayley graphs, but are vertex-transitive?

Lemma

Let $X = GC(G, S, \alpha)$ be a generalized Cayley graph on a group G with respect to the ordered pair (S, α), and let $Fix(\alpha) = \{g \in G \mid \alpha(g) = g\}$. Then $Fix(\alpha)_L \leq \mathrm{Aut}(X)$ and moreover it acts semiregularly on $V(X)$.

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Theorem

Let $X = GC(G, S, \alpha)$ be a generalized Cayley graph on a group G with respect to the ordered pair (S, α). Then there exists a non-trivial element $g \in G$, which is fixed by α. Moreover, X admits a semiregular automorphism which lies in $G_L \cap Aut(X)$.

Automorphisms of a Generalized Cayley graphs

Let $X = Cay(G, S)$ be a Cayley graph, and let $\mathrm{Aut}(G, S)$ denote the set of all automorphisms of G that fix set S, that is

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$$\mathrm{Aut}(G,S,\alpha) = \{\varphi \in \mathrm{Aut}(G) \mid \varphi(S) = S,\ \alpha\varphi = \varphi\alpha\}.$$

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Theorem

Let $X = GC(G,S,\alpha)$ be a generalized Cayley graph on a group G with respect to the ordered pair (S,α). Then $\mathrm{Aut}(G,S,\alpha) \leq \mathrm{Aut}(X)$ which fixes the vertex $1_G \in V(X)$.

Line graph of the Petersen graph

Let X be the generalized Cayley graph $GC(\mathbb{Z}_{15}, S, \alpha)$ on the cyclic group $\mathbb{Z}_{15}$ where $S = \{1, 2, 4, 8\}$ and $\alpha(x) = 11x$. Observe that the the element $3 \in \mathbb{Z}_{15}$ is fixed by α. Then the automorphism γ acting with $\gamma(x) = x + 3$ is semiregular with three orbits of length 5.

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Let us denote the vertices of X (elements of $\mathbb{Z}_{15}$) in the following way:

$$x_i = 3i + 1, \quad y_i = 3i, \quad z_i = 3i + 2; \quad i \in \{0, 1, 2, 3, 4\}.$$

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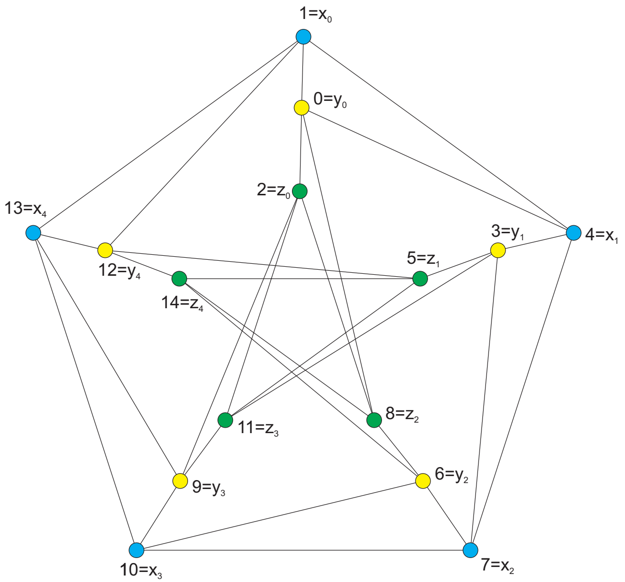

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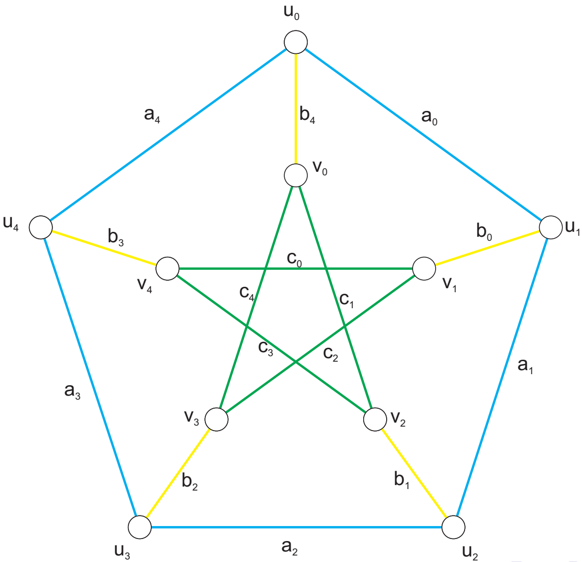

For the cyclic group $G = \mathbb{Z}_{4k}$ the mapping $\alpha\colon G \to G$ defined by the rule

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Since $2 \in G$ is of order $2k$, it gives rise to a $(2, 2k)$-semiregular automorphism in a generalized Cayley graph $GC(\mathbb{Z}_{4k}, S, \alpha)$, where S is an arbitrary subset of G satisfying the assumptions from the definition of generalized Cayley graphs.

Lemma

For a natural number $n = 4k$ let $X = GC(\mathbb{Z}_n, S, \alpha)$ be a generalized Cayley graph on $\mathbb{Z}_n$ where $\alpha(x) = (2k+1)x$. Then X is isomorphic to a bicirculant $BC_{2k}[L, M, R]$, where $L = \{s/2 \mid s \in S,\ s\text{-}even\}$, $M = \{(s-1)/2 \mid s \in S,\ s\text{-}odd\}$, and $R = k + L$.

Generalized Cayley bicirculants

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$BC_n[L, M, R]$ is a graph with vertex set
$V(X) = \{x_i \mid i \in \mathbb{Z}_n\} \cup \{y_i \mid i \in \mathbb{Z}_n\}$,
and edge set partitioned into three subsets:

$$\begin{aligned}
\mathcal{L} &= \cup_{i \in \mathbb{Z}_n} \{\{x_i, x_{i+l}\} \mid l \in L\} \quad \text{(left hand side edges)},\\
\mathcal{M} &= \cup_{i \in \mathbb{Z}_n} \{\{x_i, y_{i+m}\} \mid m \in M\} \quad \text{(middle edges - spokes)},\\
\mathcal{R} &= \cup_{i \in \mathbb{Z}_n} \{\{y_i, y_{i+r}\} \mid r \in R\} \quad \text{(right hand side edges)},
\end{aligned}$$

Generalized Cayley bicirculants

Theorem

For a natural number $k \geq 1$ *let* $n = 2((2k+1)^2 + 1)$ *and let* X *be the generalized Cayley graph* $GC(\mathbb{Z}_n, S, \alpha)$ *on the cyclic group* $\mathbb{Z}_n$ *with respect to* $S = \{\pm 2, \pm 4k^2, 2k^2 + 2k + 1\}$ *and the automorphism* $\alpha \in \mathrm{Aut}(\mathbb{Z}_n)$ *defined by the rule* $\alpha(x) = ((2k+1)^2 + 2) \cdot x$. *Then* X *is a non-Cayley vertex-transitive graph.*

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Proof.

$X \cong BC_{((2k+1)^2+1)}[\{\pm 1, \pm 2k^2\}, \{0\}, \{\pm(2k+1), \pm(2k^2+2k)\}]$.

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$X \cong BC_{((2k+1)^2+1)}[\{\pm 1, \pm 2k^2\}, \{0\}, \{\pm(2k+1), \pm(2k^2+2k)\}]$.
Prove that the set of the orbits of $(2, m)$ semiregular automorphism is system of imprimitivity for $\mathrm{Aut}(X)$. This implies that the subgroup generated by the $(2, m)$ semiregular automorphism is normal in $\mathrm{Aut}(X)$... □

Theorem

For a natural number k such that $k \not\equiv 2 \pmod 5$, $t = 2k + 1$ and $n = 20t$, the generalized Cayley graph $GC(\mathbb{Z}_n, S, \alpha)$ on the cyclic group $\mathbb{Z}_n$ with respect to $S = \{\pm 2t, \pm 4t, 5, 10t - 5\}$ and the automorphism $\alpha \in \mathrm{Aut}(\mathbb{Z}_n)$ defined by the rule $\alpha(x) = (10t + 1)x$, is a non-Cayley vertex-transitive graph.

Proof.

$$X \cong BC_{10t}[\{\pm t, \pm 2t\}, \{0, 5t - 5\}, \{\pm 3t, \pm 4t\}].$$

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Sets consisting of the left hand vertices and the right hand vertices are blocks of imprimitivity. Consider the cycles induced by the middle edges... □

It transpires that generalized Cayley graphs, specifically those associated with cyclic groups, are a rich and new source of non-Cayley vertex-transitive graphs.

Problem

Classify all generalized Cayley graphs arising from cyclic groups.

Thank you!!!