On the complexity of some packing and covering problems in certain graph classes

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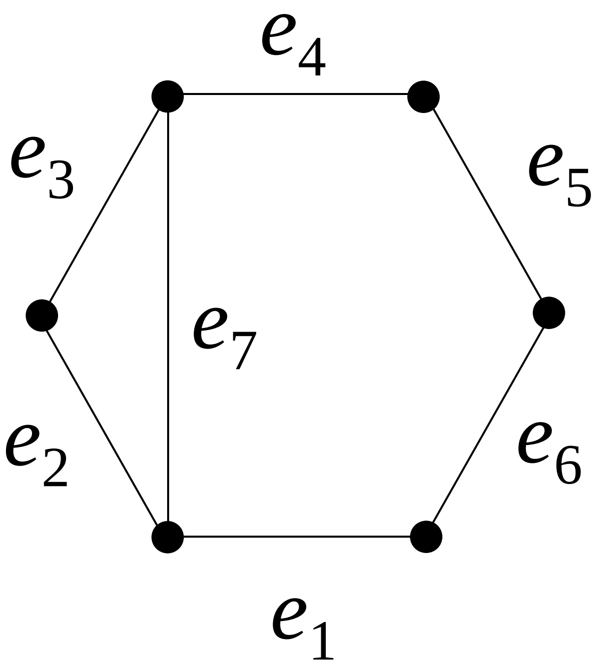
(with various coauthors)

Minimum Vertex Cover

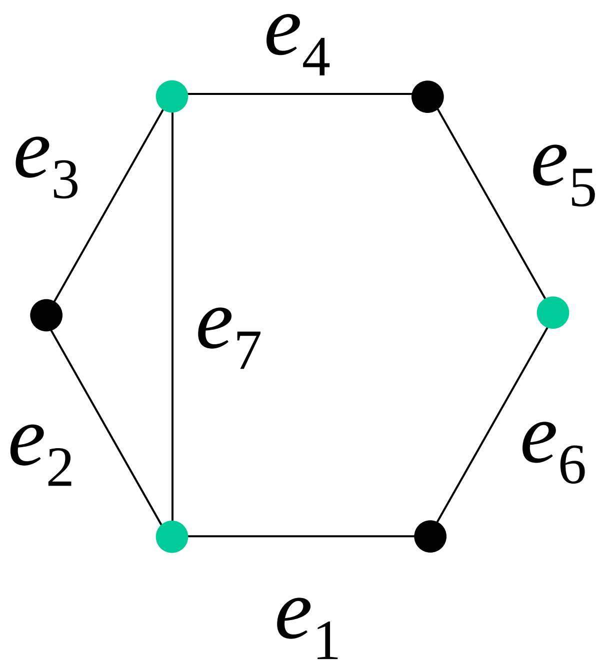
$G = (V,E)$ - a finite undirected graph.

$U \subseteq V$ is a *vertex cover* of G if for every edge e of G, U contains at least one vertex of e.

Minimum Vertex Cover



Minimum Vertex Cover



Minimum Vertex Cover

Problem [GT1] of [Garey, Johnson, 1979]:

INSTANCE: Graph $G=(V, E)$, positive integer $K \leq |V|$.

QUESTION: Is there a vertex cover of size K or less for G?

Vertex cover versus other problems

Observation.

(1) U is a vertex cover of $G \Leftrightarrow$

$V - U$ is an independent vertex set in G.

(2) U is an $\subseteq$ - maximal independent set in G
$\Leftrightarrow$ U is an independent dominating set in G.

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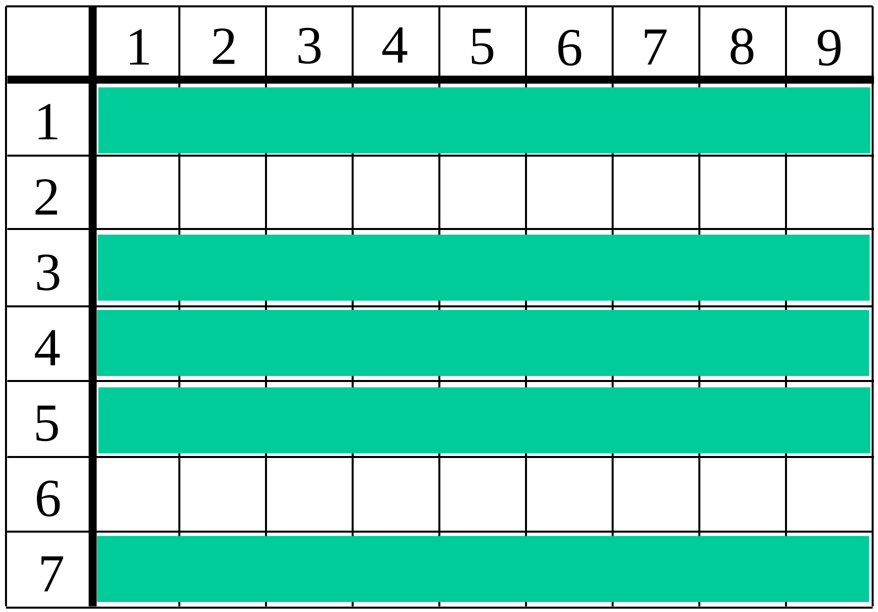
reconfigurable arrays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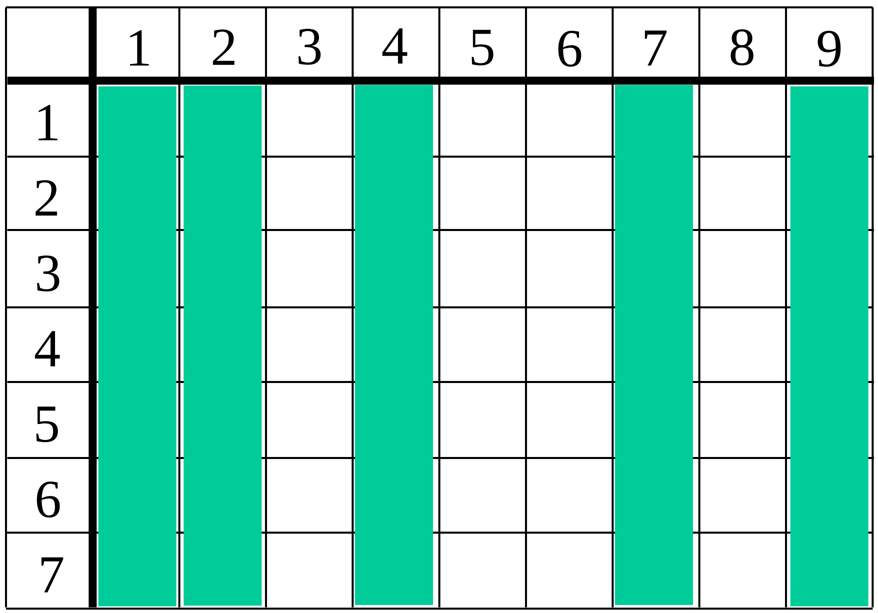
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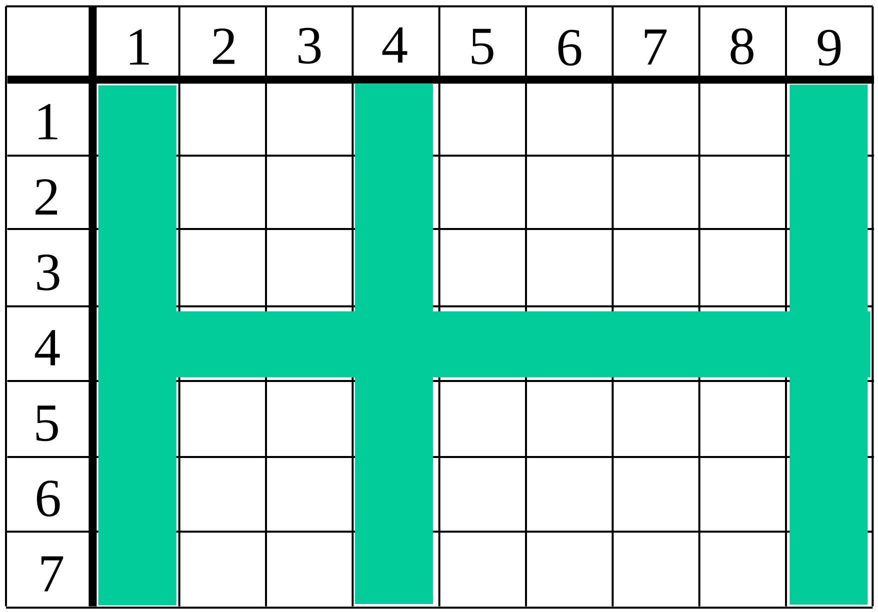
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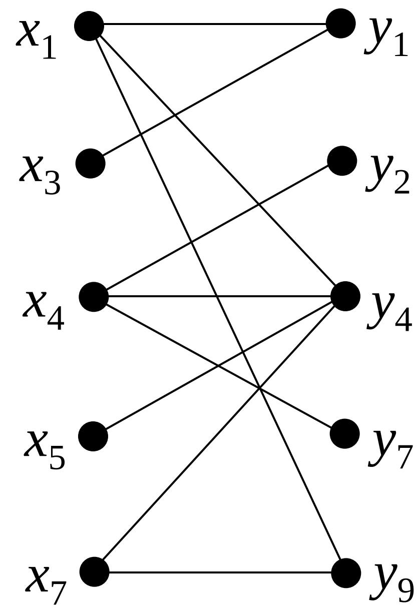

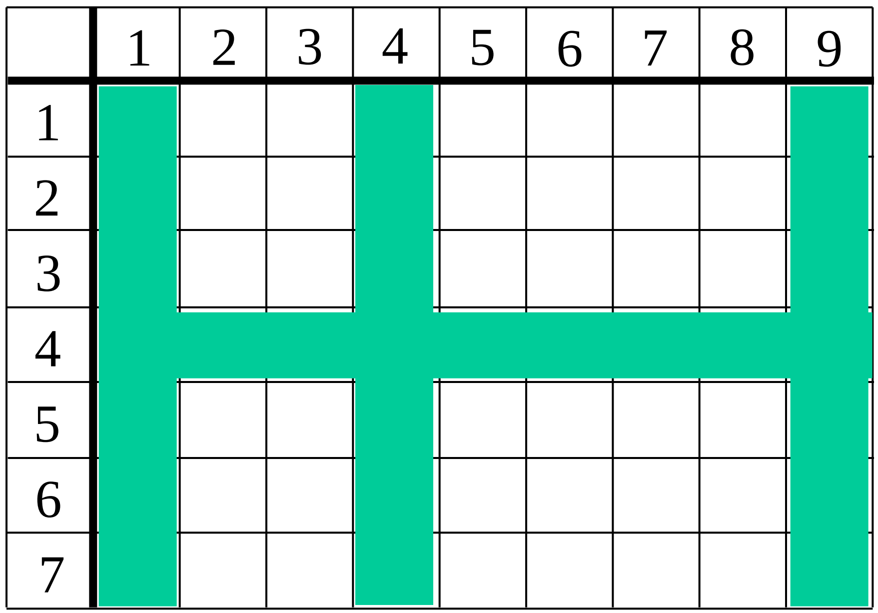
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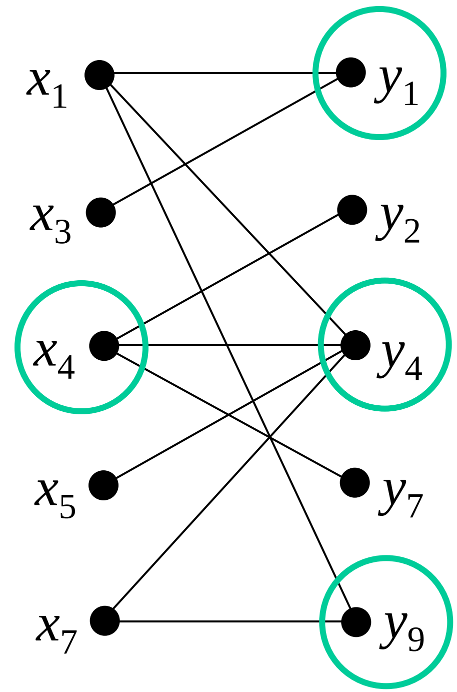
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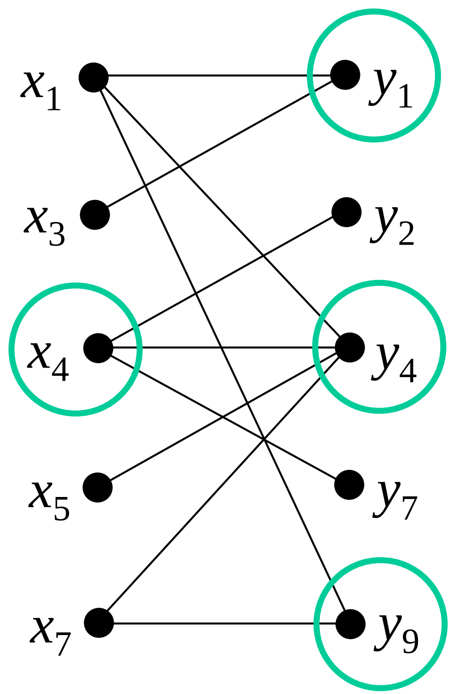

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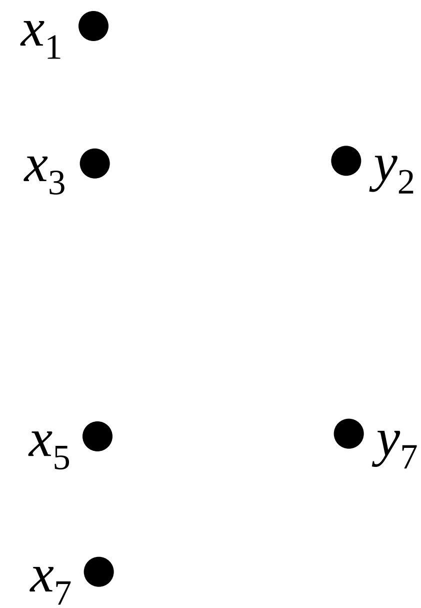

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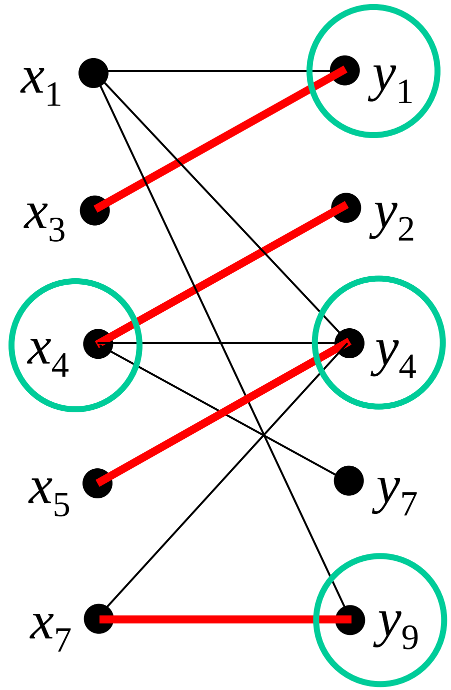
Minimum Vertex Cover

Theorem [D. König, 1930]

In a bipartite graph $B = (X, Y, E)$,

minimum size of a vertex cover =

maximum size of a matching.



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Maximum Independent Sets

$G = (V,E)$ - a finite undirected graph,

w - a *vertex weight* function on V.

$U \subseteq V$ is *stable* or *independent* if all vertices of U are pairwise nonadjacent.

$\alpha(G)$ = max *size* of a stable vertex set in G

$\alpha_w(G)$ = max *weight* of a stable vertex set in G

Maximum Independent Set

Problem [GT20] of [Garey, Johnson, 1979]:

INSTANCE: Graph $G=(V,E)$, integer K.

QUESTION: Does G contain an independent set of size K or more?

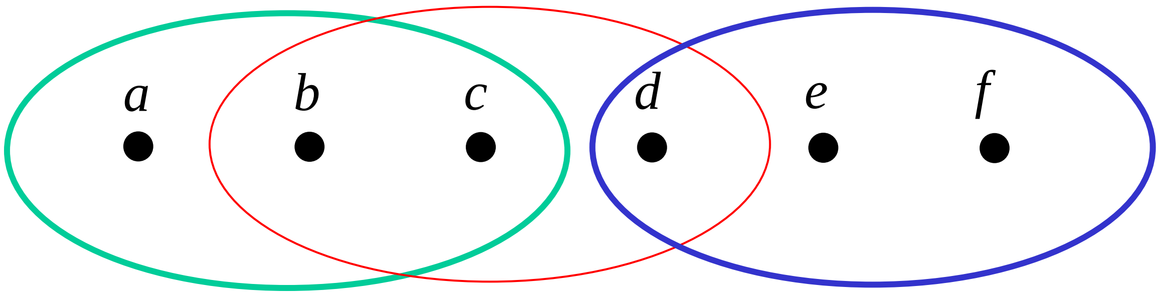
Let MIS (MWIS, resp.) denote the unweighted (vertex-weighted, resp.) problem.

Exact Cover by 3-Sets (X3C)

Problem [SP2] of [Garey, Johnson, 1979]:

INSTANCE: A finite set X with $|X| = 3q$ and a collection C of 3-element subsets of X.

QUESTION: Does C contain an *exact cover* for X, that is, a subcollection D of C such that every element of X occurs in exactly one member of D?



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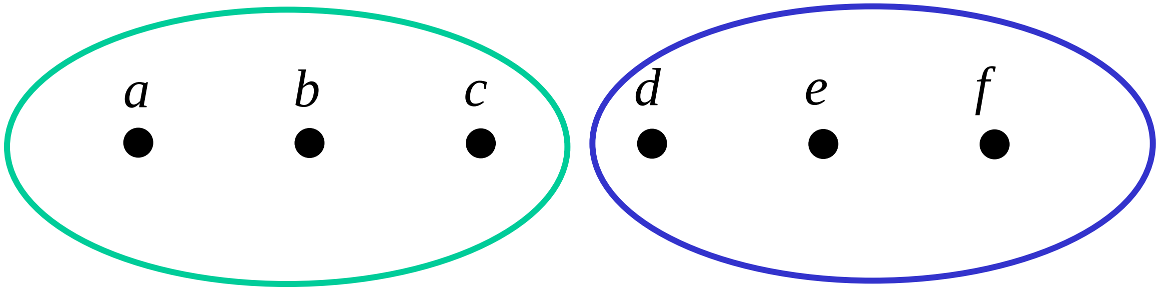

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Efficient domination

Let $G = (V,E)$ be a finite undirected graph.

A vertex v *is dominated* by itself and its neighbors, i.e., v *dominates* $N[v]$.

[Bange, Barkauskas, Slater 1988]:

D is an *efficient dominating (e.d.) set* in G if

(1) it is dominating in G and

(2) every vertex is dominated exactly once.

Efficient domination

Efficient dominating sets in G are also called *independent perfect dominating sets*.

Let $G^2 = (V, E^2)$ with $xy \in G^2$ if the distance between x and y in G is at most 2.

Fact. Let $\boldsymbol{N}(G)$ denote the closed neighbor-hood hypergraph of G. Then:

$$G^2 = L(\boldsymbol{N}(G)) \text{ holds.}$$

Efficient domination

Fact. The following are equivalent for $D \subseteq V$:

(1) D is an e.d. set in G.

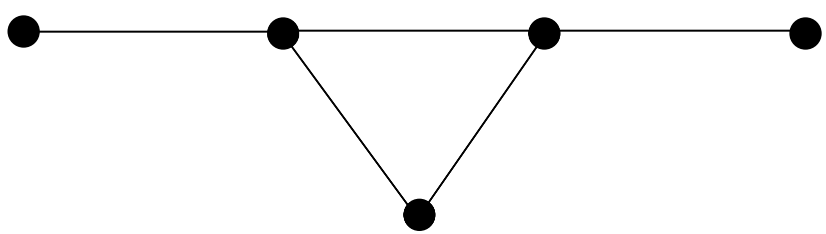
(2) D *dominating* in G and *independent* in G^2.

(3) the closed neighborhoods $N[v]$, $v \in D$, are an *exact cover of* $\boldsymbol{N}(G)$.

Corollary. G has an e.d. $\Leftrightarrow$ $\boldsymbol{N}(G)$ has an exact cover.

Efficient domination

Not every graph has an efficient dominating set!



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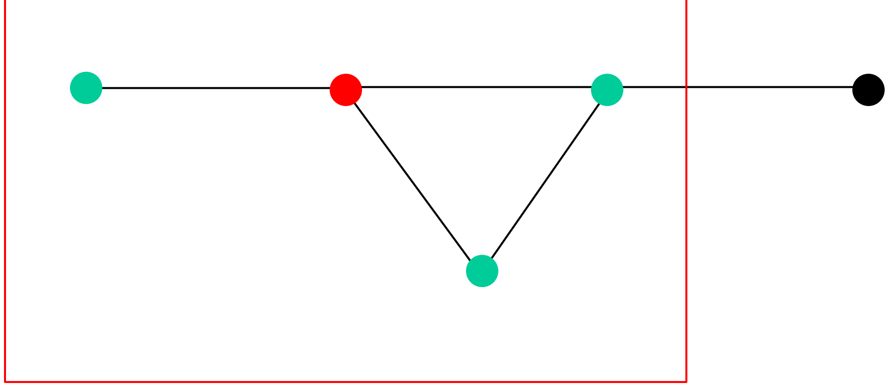

Efficient domination

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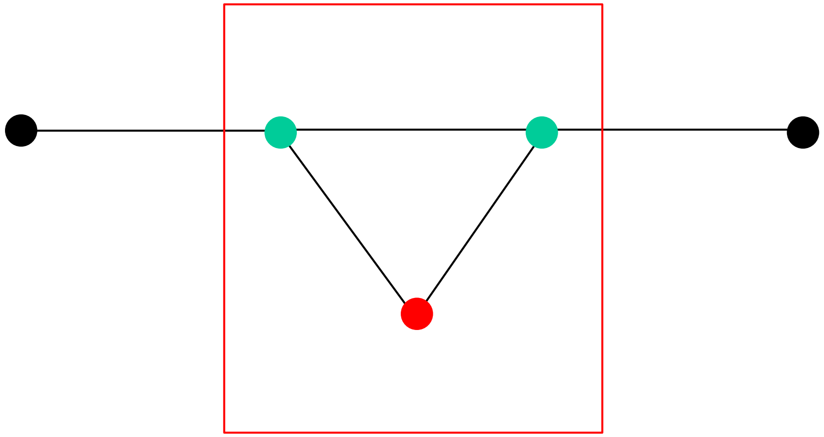
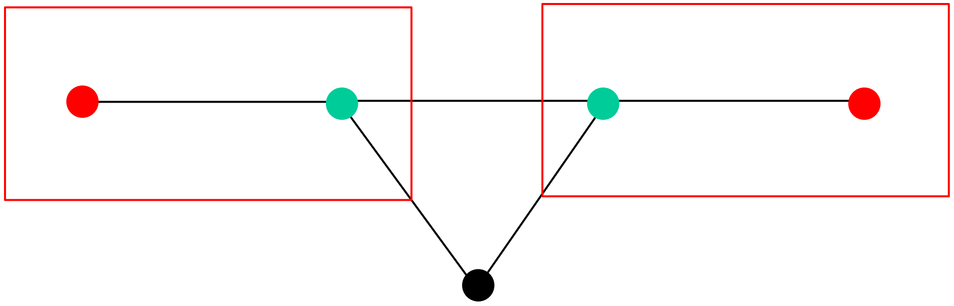
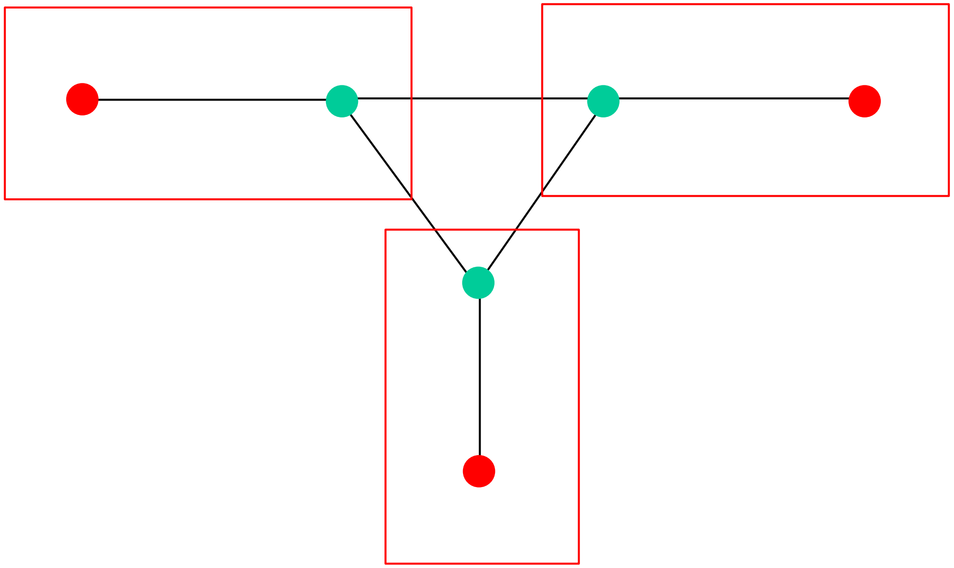


Efficient domination

Not every graph has an efficient dominating set!



Efficient domination



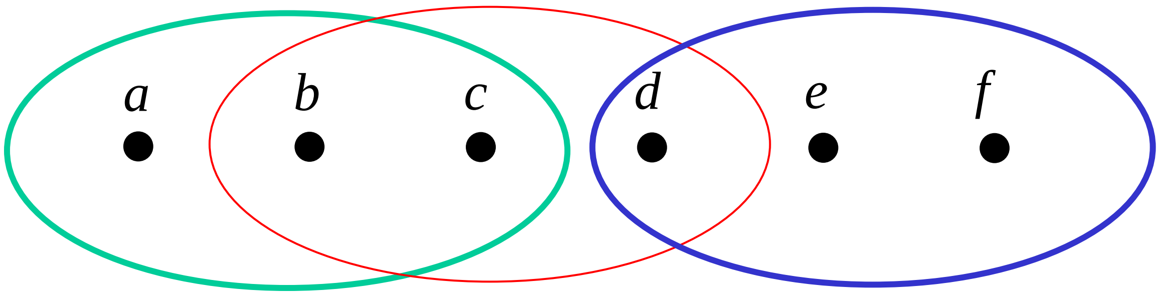
Efficient Domination (ED) Problem

INSTANCE: A finite graph $G = (V, E)$.
QUESTION: Does G have an e.d. set?

Theorem [Yen, Lee 1996]

The ED Problem is NP-complete for bipartite graphs and for chordal graphs.

Proof by simple reduction from X3C:



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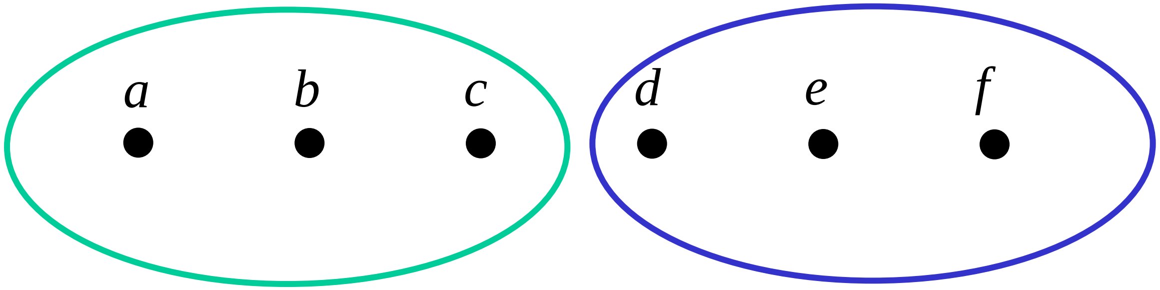

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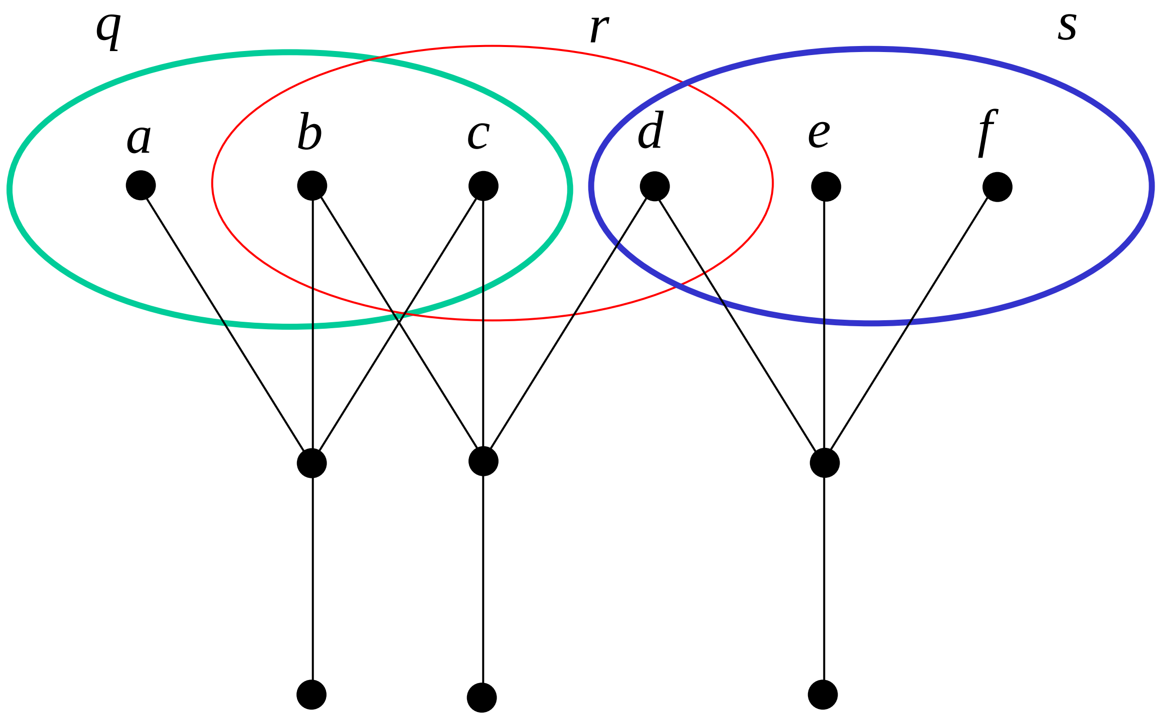
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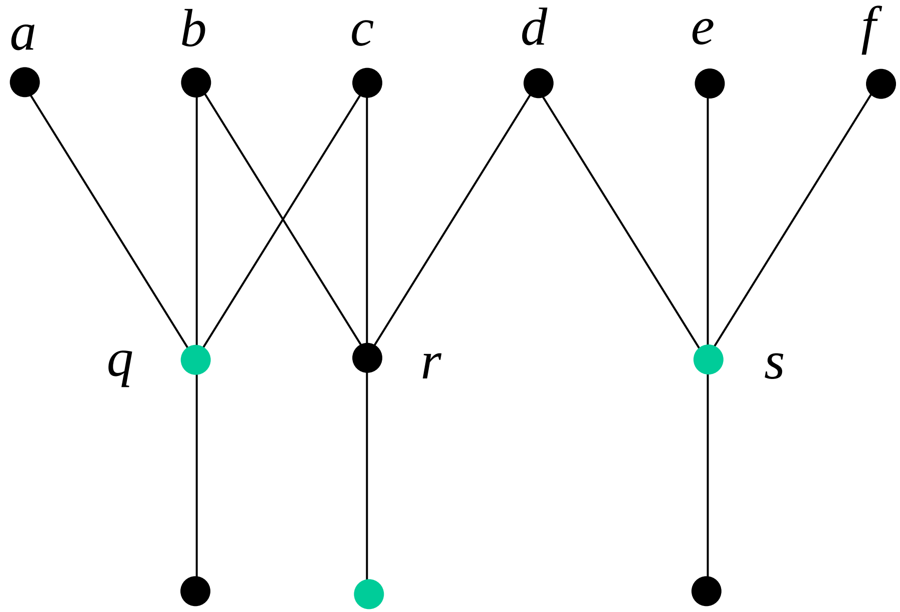
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Efficient domination

Theorem [Lu, Tang 2002]

The ED problem is NP-complete for chordal bipartite graphs.

(proof by complicated reduction from 1-in-3 3SAT)

Efficient domination

Recall: D is an efficient dominating set in G $\Leftrightarrow$ D is dominating in G and independent in G^2.

Let $w(v) := |N[v]|$. Then:

(i) D dominating in $G \Rightarrow |V| \leq w(D)$.

(ii) D independent set in $G^2 \Rightarrow w(D) \leq |V|$.

Efficient domination

Recall: $w(v) := |N[v]|$.

Fact 1 [Leitert, 2012]

D is an e.d. set in $G \Leftrightarrow D$ is a minimum weight dominating set in G with $w(D) = |V|$.

Fact 2 [Leitert; Milanič, 2012]

D is an e.d. set in $G \Leftrightarrow D$ is a maximum weight independent set in G^2 with $w(D) = |V|$.

Efficient domination

Corollary. Let $\boldsymbol{C}$ be a graph class. If the MWIS problem is solvable in polynomial time for G^2 , for all $G \in \boldsymbol{C}$**,** then the ED problem is solvable in polynomial time on $\boldsymbol{C}$.

Examples:

dually chordal graphs: squares are chordal.

AT-free graphs: squares are co-comparability.

Efficient domination

Corollary. The ED problem is solvable in polynomial time for dually chordal graphs and thus also for strongly chordal graphs.

Efficient domination

Open [Lu, Tang 2002]

Complexity of ED for convex bipartite graphs and for strongly chordal graphs.

Recall:

G strongly chordal $\Rightarrow$ G dually chordal

G convex bipartite $\Rightarrow$ G interval bigraph

$\Rightarrow$ G chordal bipartite

Efficient domination

Theorem [Bui-Xuan, Telle, Vatshelle, 2011]

If for a graph class, boolean width is at most O(log n) then the Minimum Weight Dominating Set problem can be solved in polynomial time.

Theorem [Keil, 2012] Boolean width of interval bigraphs is at most 2 log n.

Efficient domination

Corollary.

For interval bigraphs, the ED problem can be solved in polynomial time.

Recall:

G convex bipartite $\Rightarrow$ G interval bigraph

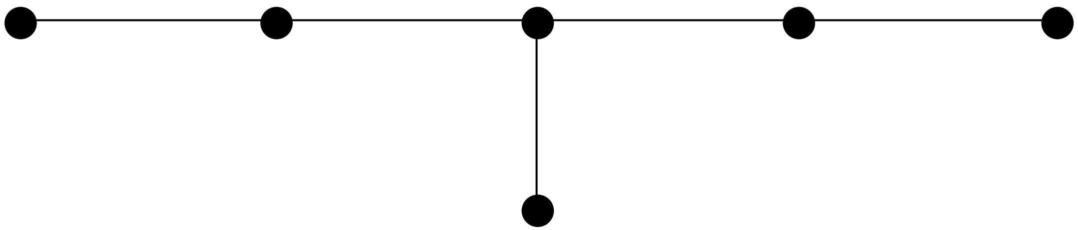
Efficient edge domination

[Grinstead, Slater, Sherwani, Holmes, 1993]:

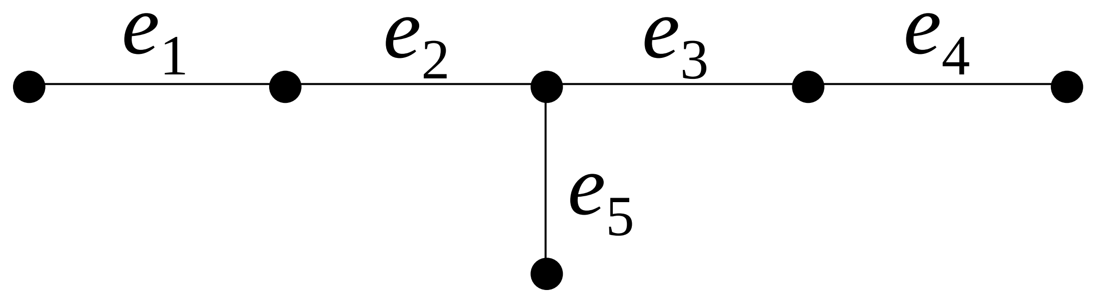
$M \subseteq E$ is an *efficient edge dominating* (*e.e.d.*) *set* in G if M is dominating in $L(G)$ and every edge of E is dominated exactly once in $L(G)$ (that is, M is an efficient dominating set in $L(G)$).


Vadim Lozin

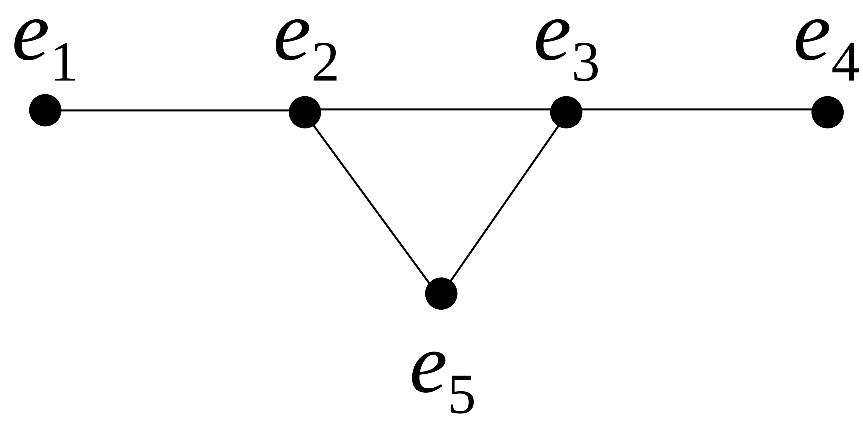
Not every graph (not every tree !) has an efficient edge dominating set:

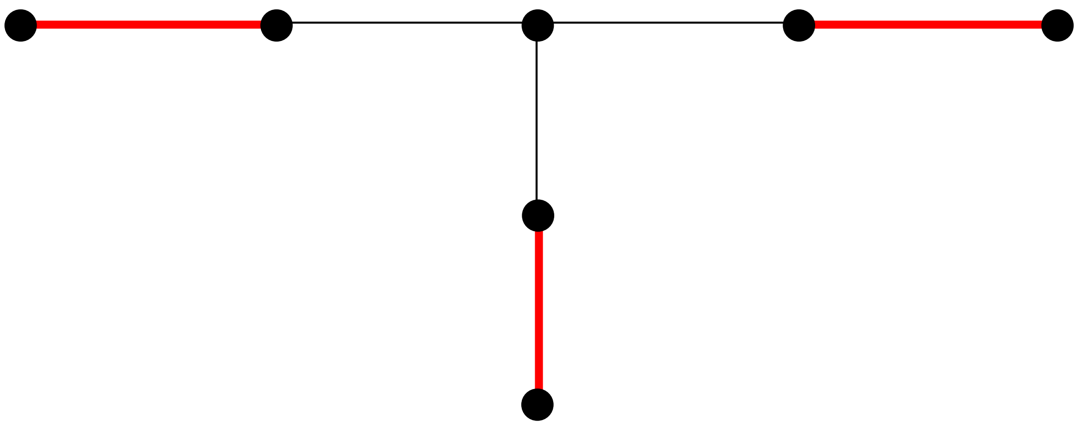


G:



$L(G)$:





Efficient Edge Domination (EED) Problem

INSTANCE: A finite graph $G = (V, E)$.
QUESTION: Does G have an e.e.d. set?

Theorem [Grinstead, Slater, Sherwani, Holmes, 1993]

The EED Problem is NP-complete.

Efficient edge domination

Efficient edge dominating sets are also called *dominating induced matchings* (*d.i.m.*) :

Fact. M is an e.e.d. in graph $G = (V,E) \Leftrightarrow$

(1) M is an *induced matching* in G (that is, pairwise distance of edges in M at least 2),

(2) every edge in E is intersected by exactly one edge from M.

Efficient edge domination

Theorem [Lu, Tang 1998, Lu, Ko, Tang 2002]

EED is NP-complete for bipartite graphs, and is solvable in linear time for bipartite permutation graphs, generalized series-parallel graphs and for chordal graphs.

Efficient edge domination

Theorem [Lu, Tang 1998, Lu, Ko, Tang 2002]

EED is NP-complete for bipartite graphs, and is solvable in linear time for bipartite permutation graphs, generalized series-parallel graphs and for chordal graphs.

Open [Lu, Ko, Tang 2002]

Complexity of EED for weakly chordal graphs and for permutation graphs.

Efficient edge domination

Theorem [Cardoso, Lozin 2008]

EED is NP-complete for (very special) bipartite graphs, and is polynomial time solvable for claw-free graphs.

Efficient edge domination

Open [Cardoso, Korpelainen, Lozin 2011]

Complexity of EED for

- P_k–free graphs, $k > 4$
- chordal bipartite graphs
- weakly chordal graphs

Efficient edge domination

Theorem [B., Hundt, Nevries 2009, LATIN 2010] The EED problem is solvable in

- linear time for chordal bipartite graphs,
- polynomial time for hole-free graphs, and
- is NP-complete for planar bipartite graphs with maximum degree 3.

Efficient edge domination

Theorem [B., Mosca, ISAAC 2011]

EED in linear time for P_7 -free graphs in a robust way.

Efficient edge domination

Theorem [B., Mosca, ISAAC 2011]

EED in linear time for P_7 -free graphs in a robust way.

EED in Monadic Second Order Logic:

Fact. $G = (V, E)$ has an e.e.d. $\Leftrightarrow$

$$\exists E' \subseteq E \;\; \forall e \in E \;\; \exists!\, e' \in E' \; (e \cap e' \neq \emptyset)$$

Efficient edge domination

Recall: D is an efficient edge dominating set in $G \Leftrightarrow D$ is dominating in $L(G)$ and independent in $L(G)^2$, i.e., D is an e.d. set in $L(G)$.

Let $w(e) := |N[e]|$ (neighborhood w.r.t. $L(G)$).

Fact. M is an efficient edge dominating set in $G \Leftrightarrow M$ is a maximum weight independent set in $L(G)^2$ with $w(M) = |E|$.

Squares of Line Graphs

- G chordal $\Rightarrow L(G)^2$ chordal [Cameron 1989]
- G circular-arc $\Rightarrow L(G)^2$ circular-arc [Golumbic, Laskar 1993]
- G co-comparability $\Rightarrow L(G)^2$ co-comparability [Golumbic, Lewenstein 2000]
- G weakly chordal $\Rightarrow L(G)^2$ weakly chordal [Cameron, Sritharan, Tang 2003]
- stronger result for AT-free graphs [J.-M. Chang 2004]

Efficient edge domination

Recall: If the MWIS problem is solvable in polynomial time for the squares of the line graphs of all graphs in ***C*** then the EED problem is solvable in polynomial time on ***C***.

Corollary.

EED in polynomial time for weakly chordal graphs and for permutation graphs.

ED for hypergraphs

$H = (V,E)$ - a finite hypergraph.

$D \subseteq V$ is an *e.d. set* in H if D is an e.d. set in 2sec(H).

Theorem. The ED problem is NP-complete for α–acyclic hypergraphs, and is solvable in polynomial time for hypertrees.

EED for hypergraphs

$H = (V,E)$ - a finite hypergraph.

$M \subseteq E$ is an *e.e.d. set* in H if M is an e.e.d. set in $L(H)$.

Theorem. The EED problem is solvable in polynomial time for α–acyclic hypergraphs, and is NP-complete for hypertrees.

Maximum induced matchings for hypergraphs

$H = (V,E)$ - a finite hypergraph.

$M \subseteq E$ is an *induced matching* in H if M is an independent node set in $L(H)^2$.

Theorem. The MIM problem is solvable in polynomial time for α–acyclic hypergraphs, and is NP-complete for hypertrees.

Exact Cover for hypergraphs

Theorem. The Exact Cover problem is NP-complete for α–acyclic hypergraphs, and is solvable in polynomial time for hypertrees.

	chordal	dually chordal	acyclic hyp.	hypertrees
ED	NP-c. [ ]	lin.	NP-c.	pol.
EED	lin. [ ]	lin.	pol.	NP-c.
MIM	pol. [ ]	NP-c.	pol.	NP-c.
XC			NP-c.	pol.

Thank you for your attention!

Thank you for your attention!

Thank you for your attention!

Exact Cover

Let $H = (V, E)$ be a hypergraph, C a collection of hyperedges, and let $w(e) := |e|$. If the edges in C are pairwise disjoint then $w(C) \leq |V|$.

Fact. The following are equivalent:

- C is an exact cover of V.

- edges in C pairwise disjoint and $w(C)=|V|$.

- C maximum independent in $L(H)$, $w(C)=|V|$.

Exact Cover

Corollary. Exact Cover is solvable in polynomial time for every class ***C*** of hypergraphs for which MWIS is solvable in polynomial time on the line graphs of ***C***.

Example. Exact Cover is solvable in polynomial time for *hypertrees*: Their line graphs are chordal.

Hypertrees

A hypergraph H is a *hypertree* if there is a tree T such that every hyperedge of H induces a connected subgraph in T.

Theorem [Duchet, Flament, Slater 1976] H is a hypertree $\Leftrightarrow$ H is Helly and $L(H)$ is chordal.

Dually chordal graphs

Let $N(G)$ denote the *closed neighborhood hypergraph* of graph G.

A graph G is *dually chordal* if $\boldsymbol{N}(G)$ is a hypertree.

Fact.

$G^2 = L(\boldsymbol{N}(G))$.

Dually chordal graphs

Theorem [B., Dragan, Chepoi, Voloshin1994]

G is dually chordal $\Leftrightarrow$ G^2 is chordal and $\boldsymbol{N}(G)$ has the Helly property $\Leftrightarrow$ its clique hypergraph is a hypertree.

(and various other characterizations in [BDCV 1994], [Szwarcfiter, Bornstein1994], [Gutierrez, Oubina 1996] …

Dually chordal graphs

Theorem [B., Dragan, Chepoi, Voloshin1994]

G is strongly chordal $\Leftrightarrow$ G is hereditarily dually chordal.

Thank you for your attention!

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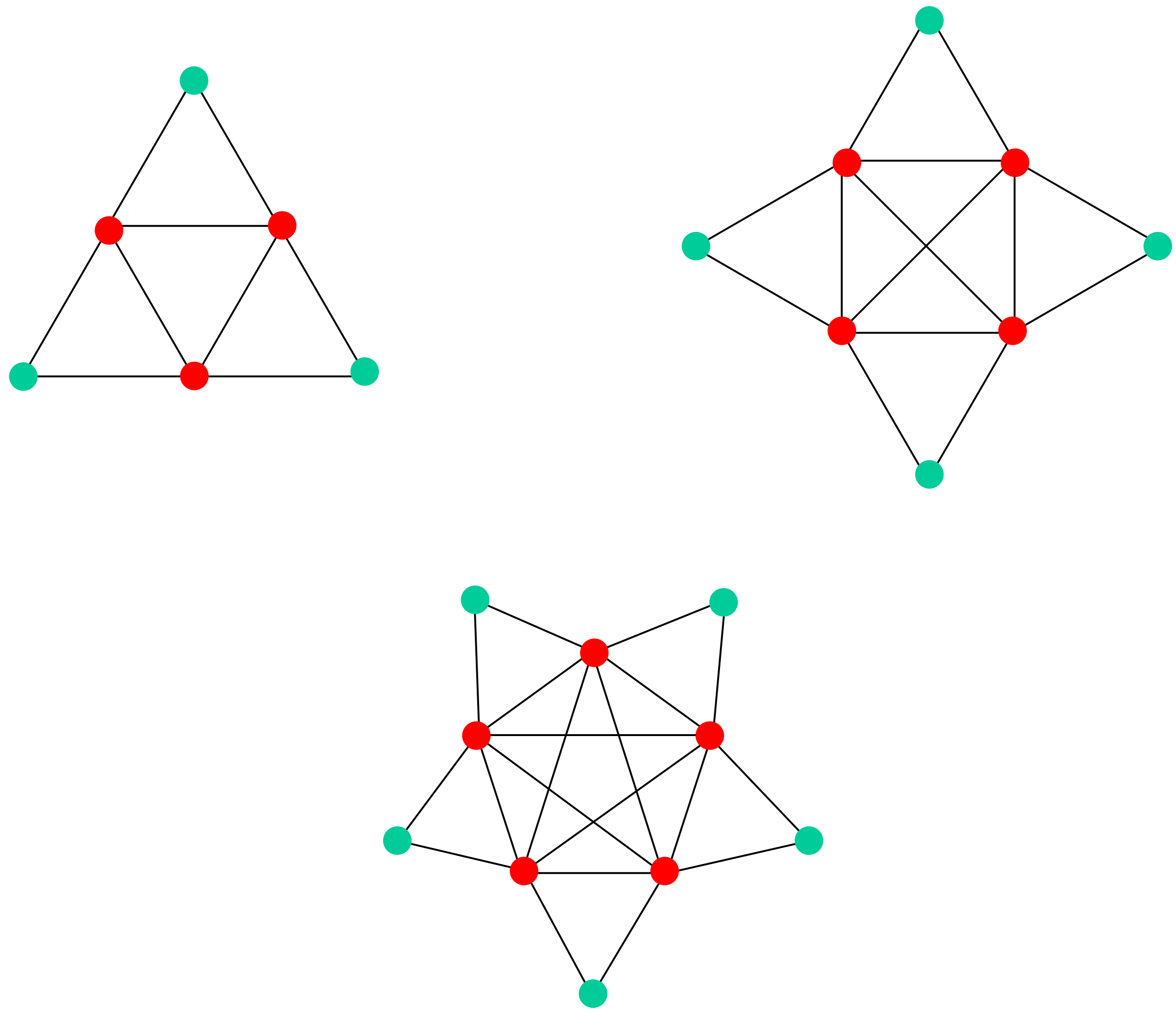


Strongly chordal graphs

G is *strongly chordal* if G is sun-free (i.e., S_k – free for any $k \geq 3$) and chordal.

Theorem.

G is strongly chordal $\Leftrightarrow$ it is hereditarily dually chordal.

(and many other characterizations …)

Theorem [Lubiw 1982; Dahlhaus, Duchet 1987; Raychaudhuri 1992] For every $k \geq 2$:

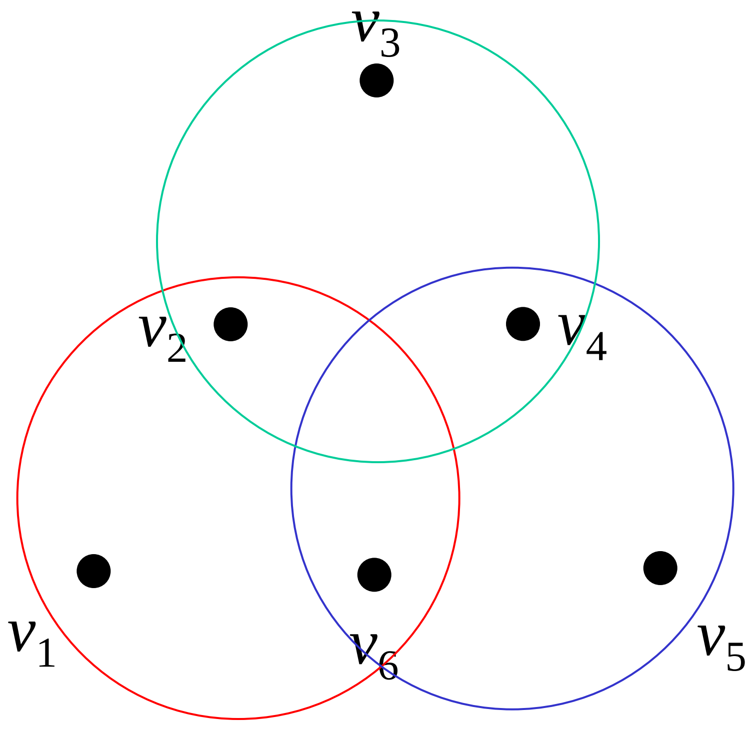
G strongly chordal $\Rightarrow G^k$ strongly chordal.

Tree Structure of Hypergraphs

Let $H=(V, E)$ be a hypergraph.

T is a *join tree* of H if E is the set of nodes of T and for all vertices $v \in V$, the occurencies of v in nodes of T form subtrees of T.

H is α–*acyclic* if H has a join tree.



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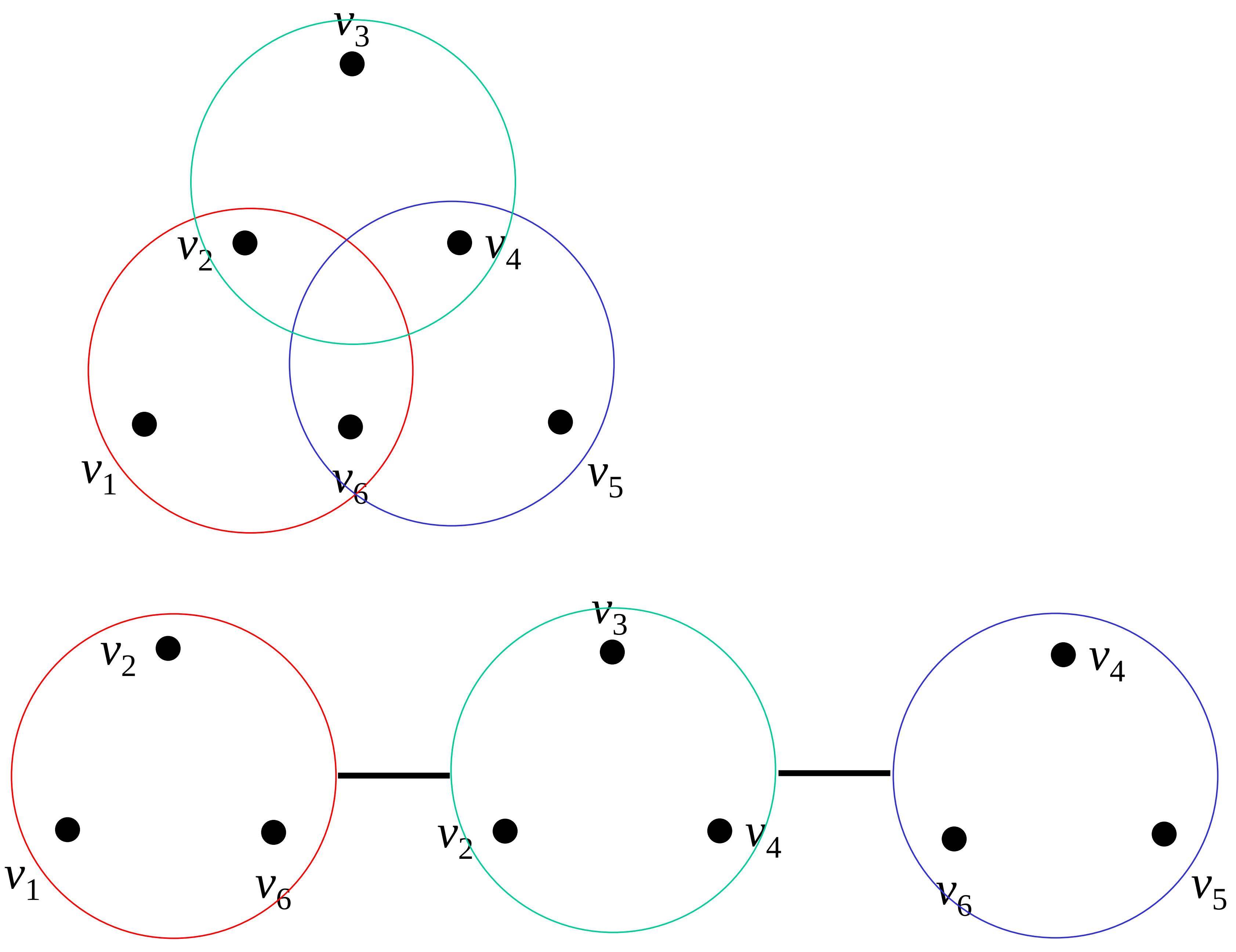

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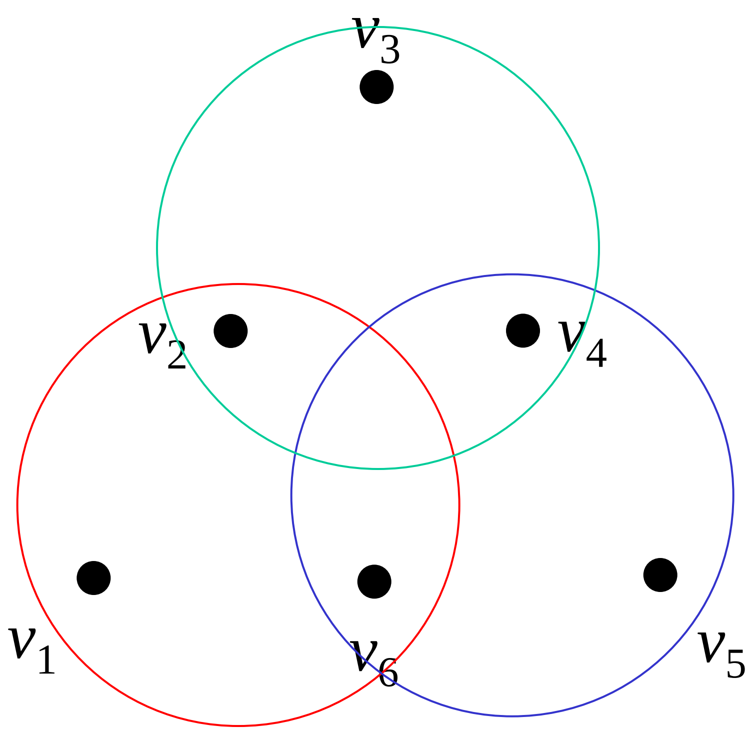
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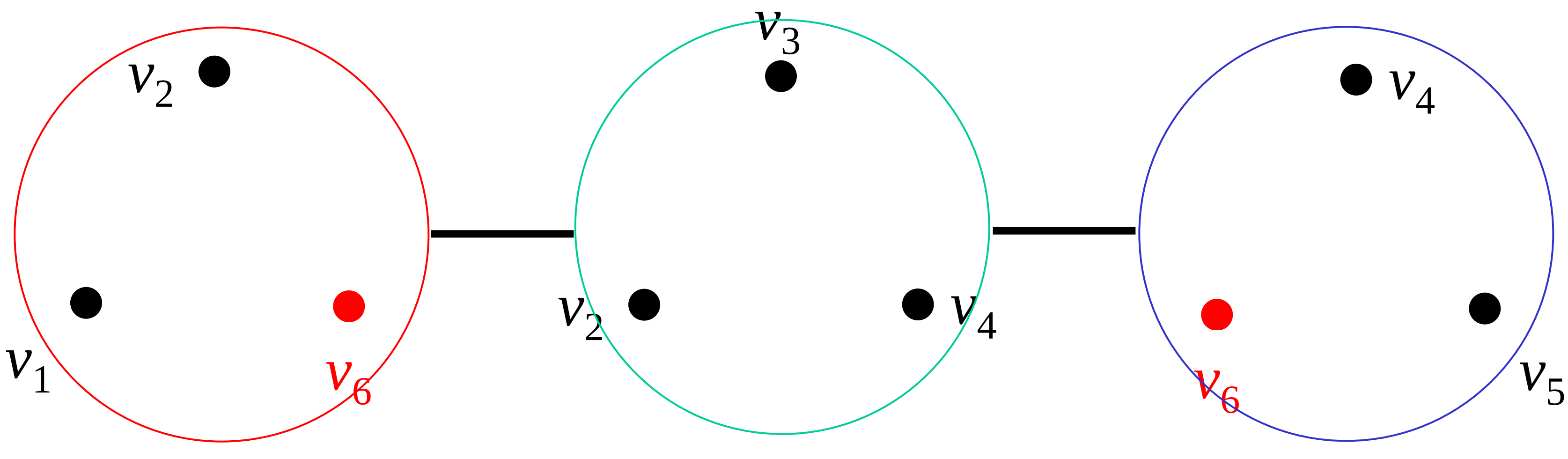
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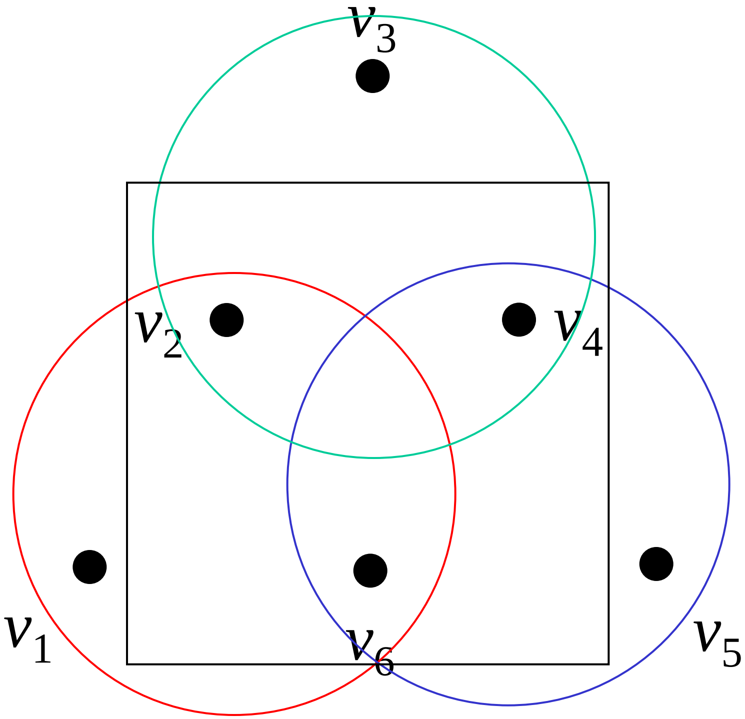
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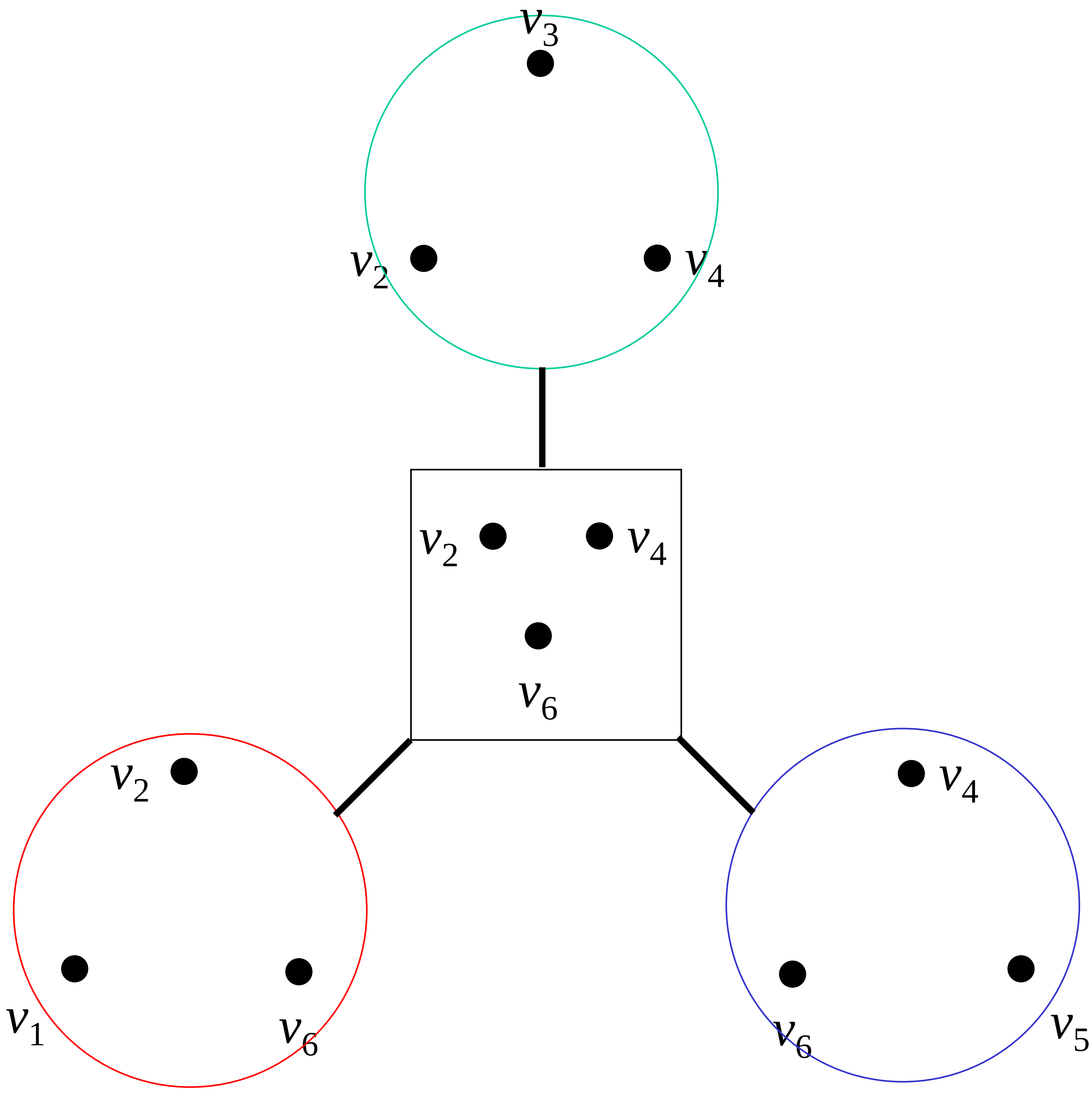

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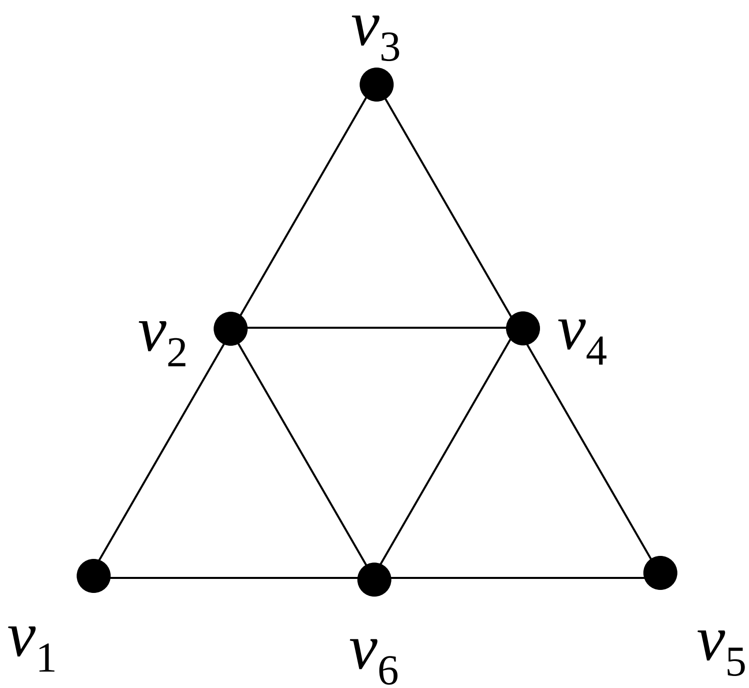
v_4
v_2
v_4
v_6
v_2
v_1
v_6
v_4
v_6
v_5

Chordal graphs

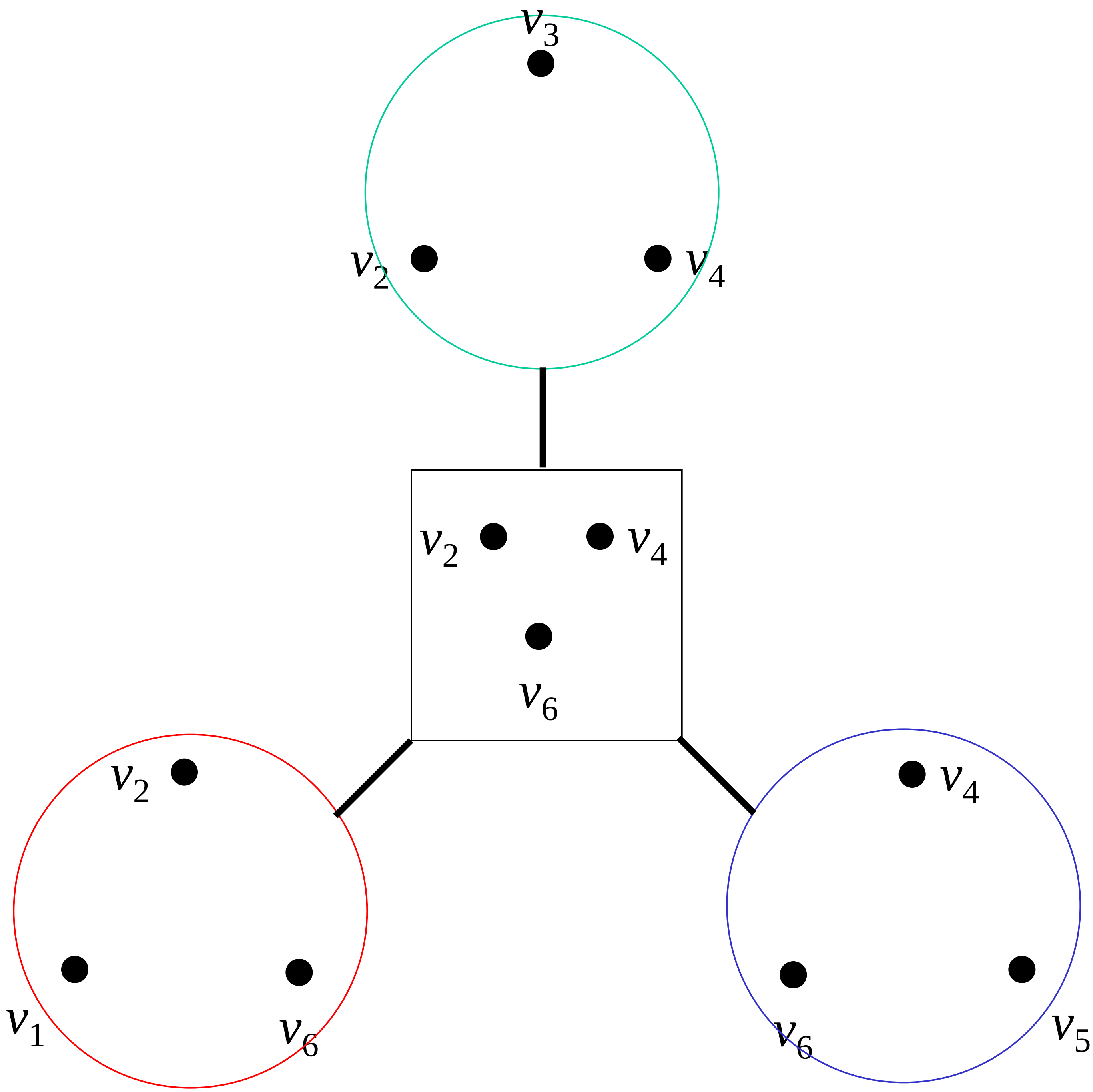
Theorem [Walter, Gavril, Buneman 1972]

G is chordal $\Leftrightarrow$ G is the intersection graph of subtrees of a tree.

Corollary. G is chordal $\Leftrightarrow$ the hypergraph of its $\subseteq$-maximal cliques has a join tree (which is called *clique tree* for graphs).



v_3
v_2
v_4
v_1
v_6
v_5



v_3
v_2
v_4
v_2
v_4
v_6
v_2
v_1
v_6
v_4
v_6
v_5

Tree Structure

Let H=(V, E) be a hypergraph.

H is *conformal* if every maximal clique of its 2-section graph is contained in a hyperedge.

Theorem [Duchet, Flament, Slater 1976]

H is α–acyclic $\Leftrightarrow$ H is conformal and its 2-section graph is chordal.

Tree Structure

Fact. H is a hypertree $\Leftrightarrow$ its dual is α–acyclic.

Theorem [Duchet, Flament, Slater 1976]

H is a hypertree $\Leftrightarrow$ *H* is Helly and its line graph is chordal.