Bourgin-Yang theorems for the p-toral groups

from **Wacław Marzantowicz** at

to **Univerza na Primorskem**

Thursday 09.06.2016; **under the spirit of**

In 1933 S. Ulam posed and K. Borsuk (Charles Badger) showed that if $n > m$
then **it is impossible** to map $f : S^n \to S^m$

$$\text{preserving symmetry: } f(-x) = -f(x) \ .$$

Next in 1954-55, C. T. Yang, and D. Bourgin, showed that if $f : \mathbb{S}^n \to \mathbb{R}^{m+1}$ preserves this symmetry then

$$\dim f^{-1}(0) \geq n - m - 1.$$

We will present versions of the latter for some other groups of symmetries and also discuss the case $n = \infty$.

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We estimate the dimension of set $Z_f = f^{-1}\{0\}$ in terms of $\dim V$ and $\dim W$, if G is the torus $\mathbb{T}^k$, the p-torus $\mathbb{Z}_p^k$, or the cyclic group $\mathbb{Z}_{p^k}$, p-prime.

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Finally, we show that for any p-toral group:
$e \hookrightarrow \mathbb{T}^k \hookrightarrow G \to \mathcal{P} \to e$, P a finite p-group,
and a G-map $f : S(V) \to W$, with $\dim V = \infty$ and $\dim W < \infty$, we have $\dim Z_f = \infty$.

Proof of the Borsuk-Ulam

Suppose that $f : S(\mathbb{R}^{n+1}) \stackrel{\mathbb{Z}_2}{\to} S(\mathbb{R}^{m+1})$ and $n > m$ is such a map.

Let $\iota : \mathbb{R}^{m+1} \subsetneq \mathbb{R}^{n+1}$ natural embedding.

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i) and ii) lead to a contradiction. $\square$

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d) *The Brouwer fixed-point theorem.*

Example

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e) The case $n = 1$: there always $\exists$ a pair of opposite points on the earth's equator with the same temperature.

Necklace splitting problem

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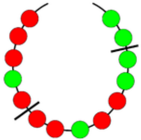
Then the necklace splitting problem: find a partition of the necklace into k parts (not necessarily contiguous), each with exactly a_i beads of colour i; called a k-split.



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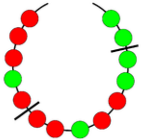

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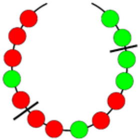

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Imre Bárány (1978) gave a simple proof, using the Borsuk-Ulam t. and a lemma of David Gale.

Theorem (Ham sandwich theorem for measures)

Let $\mu_1, \mu_2, ..., \mu_d$ be finite Borel measures on $\mathbb{R}^d$ such that every hyperplane has measure 0 for each of the μ_i (we refer to such measures as mass distributions).
Then there exists a hyperplane h such that $\mu_i(h_+) = \frac{1}{2}\mu_i(\mathbb{R}^d)$ for $i = 1, 2, .., d$, where h_+ denotes one of the half-spaces defined by h.

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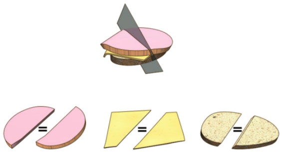
Theorem (Ham sandwich theorem for point sets)

Let $A_1, A_2, ..., A_d \subset \mathbb{R}^d$ be finite point sets. Then there exists a hyperplane h that simultaneously bisects $A_1, A_2, ..., A_d$.

The idea of proof is very simple: replace the points of A_i by tiny balls and apply the ham sandwich theorem for measures. But there are some subtleties along the way.

Borsuk-Ulam



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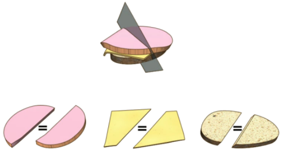
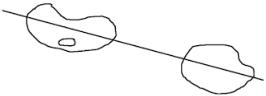


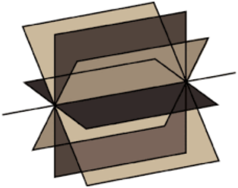
The ham-sandwich theorem in $\mathbb{R}^3$ with three ingredients

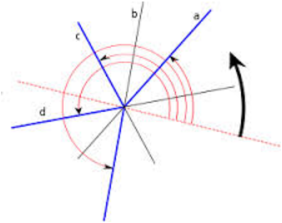


The pancakes theorem in $\mathbb{R}^2$ with two ingredients = each of distinct cake.

Bourgin-Yang

The pancakes theorem in $\mathbb{R}^3$: two pancakes $\subset \mathbb{R}^2 \subset \mathbb{R}^3$





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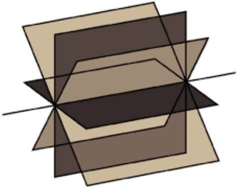
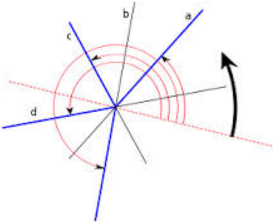
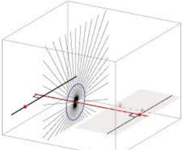




Each plane cutting $\mathbb{R}^2$ along the line of solution in $\mathbb{R}^2$ gives a solution in $\mathbb{R}^3$. Their unit normal vectors form $Z_f = \mathbb{S}^1 \subset \mathbb{S}(\mathbb{R}^3)$



Borsuk-Ulam: Hundreds of generalizations & several applications

See the article of Steinlein [25] for generalizations

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Fibrewise setting: $f : S(E) \subset E \longrightarrow E'$ over B.

1981- Jaworowski [12];

1988 - Dold [10] for $G = \mathbb{Z}_2$;

1989 - Nakaoka [23] for $G = \mathbb{Z}_2$, $\mathbb{Z}_p$ and S^1;

1990 - Izydorek and Rybcki [11] for $G = \mathbb{Z}_p$;

1995 - Kosta-Mramor [19] for Banach vector bundles;

2007 - de Mattos and dos Santos [9] for $G = \mathbb{Z}_p$, and a product of spheres;

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Applications:

Nonlinear Analysis: See *Topological Methods for Variational Problems with Symmetries* of T. Bartsch [3]

Combinatorics: See *Using the Borsuk-Ulam Theorem* of J. Matousek [18]

"Classical" Bourgin-Yang theorem

– Bourgin-Yang theorem for the mappings of spheres representation $f : S(V)\stackrel{G}{\rightarrow}W$

2012 - W. M. , de Mattos and dos Santos [16]
for $G = \mathbb{Z}_{p^k}$, a use of equivariant K-theory;

2013-2015 - W. M., de Mattos and dos Santos [17]
for $G = (\mathbb{Z}_p)^k$, $(\mathbb{Z}_2)^k$, $(S^1)^k$, a use of Borel cohomology;

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– 2016 - W.M., Błaszczyk, Singh [4] for a general setting: X, Y more general,
  a combination of methods.

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It studies

$$\dim A(f) = \{x \in X : \mid f(x) = f(gx), \text{ for all } g \in G\}$$

for a map (not equivariant in general) $f : X \to Y$ of G-spaces X and Y.

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There are relations but not direct - we will not discuss the latter.

Idea od proof

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If $f : S(\mathbb{R}^{n+1}) \xrightarrow{\mathbb{Z}_2} \mathbb{R}^{m+1}$ *then* $\dim Z_f \geq n - m - 1.$

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X a free G-space (metric, CW-complex),
$\phi : X/G \to BG = \mathbb{R}P(\infty)$ a map classifying $p : X \to X/G$.
$\gamma \in H^1(BG; F_2)$, here $H^*(BG; F_2) = F_2[\gamma]$.
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$i_G(S^n \ldots) = n + 1$; $i_G(X) \leq$ cohdim $X + 1$

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$$n + 1 = i_G(S(\mathbb{R}^{n+1})) \leq i_G(V) + i_G(U)\,,$$

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$$\text{because } f : V \to \mathbb{R}^{m+1} \setminus \{0\} \sim S(\mathbb{R}^{m+1}).$$

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$$\mathrm{coh.dim} Z_f \geq n - m - 1\,. \square$$

Notation

$\dim X$: the covering dimension of X and $\mathrm{coh.dim} X$ the cohomological dimension of a space X, i.e.,

$$\mathrm{coh.dim} X = \max\{n \mid \check{H}^n(X) \neq 0\}$$

where $\check{H}^n(-)$ denotes the Čech cohomology with coefficients $\mathbb{F} = \mathbb{Z}_p$ or $\mathbb{F} = \mathbb{Q}$, depending on whether $G = \mathbb{Z}_p^k$ or $G = \mathbb{T}^k$.

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We have $\mathrm{coh.dim} X \leq \dim X$.

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$$\mathrm{coh.dim}X = \max\{n \mid \check{H}^n(X) \neq 0\}$$

where $\check{H}^n(-)$ denotes the Čech cohomology with coefficients $\mathbb{F} = \mathbb{Z}_p$ or $\mathbb{F} = \mathbb{Q}$, depending on whether $G = \mathbb{Z}_p^k$ or $G = \mathbb{T}^k$.

We have $\mathrm{coh.dim}X \leq \dim X$.

Also, $H_*(-)$, $H^*(-)$ ($\tilde{H}_*(-)$, $\tilde{H}^*(-)$) the (reduced) singular (co)homology with coefficients $\mathbb{F} = \mathbb{Z}_p$ or $\mathbb{F} = \mathbb{Q}$, depending on whether $G = \mathbb{Z}_p^k$ or $G = \mathbb{T}^k$.

Recall that for $G = \mathbb{Z}_p^k$, with p prime odd, and $G = \mathbb{T}^k$ $\forall$ nontrivial irreducible orthogonal representation
is even dimensional and admits the complex structure,
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For a G-map $f : S(V) \to W$ we study the set

$$Z_f := f^{-1}(0) \subset S(V)$$

Let $\mathcal{A}$ be a set of G-spaces, h^* a multiplicative equivariant cohomology theory, and $I \subseteq h^*(\mathrm{pt})$ an ideal.

Definition

The $(\mathcal{A}, h^*, I)$-*length* of a G-space X is defined to be the smallest integer $k \geq 1$ such that there exist $A_1, \ldots, A_k \in \mathcal{A}$ with the property that for any $\alpha_i \in I \cap \ker\big[h^*(\mathrm{pt}) \to h^*(A_i)\big]$, $1 \leq i \leq k$,

$$p_X^*(\alpha_1) \smile \cdots \smile p_X^*(\alpha_k) = 0,$$

where $p_X : X \to \mathrm{pt}$.

Theorem ([3, Theorem 4.7])

The length has the following properties:

(1) *If there $\exists$ an h^*-functorial G-equivariant map $X \to Y$, then $\ell(X) \leq \ell(Y)$.*

(2) *Let A, $B \subseteq X$ be G-invariant subspaces such that*
$$h^*(X, A) \times h^*(X, B) \overset{\smile}{\longrightarrow} h^*(X, A \cup B)$$
is defined. If $A \cup B = X$, then $\ell(X) \leq \ell(A) + \ell(B)$.

(3) *If $h^* = H^*_G$, I is noetherian and X is paracompact, then any $GA = A$, $\mathrm{cl}\, A = A \subseteq X$ has an open G-neighborhood $\mathcal{U} \subseteq X$ such that $\ell(\mathcal{U}) = \ell(A)$.*

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Depending on the group G, we set:

(1) if $G = (\mathbb{Z}_2)^k$: $h^* = H^*_G(-;\mathbb{Z}_2)$, $I = H^*_G(\mathrm{pt};\mathbb{Z}_2)$,

(2) if $G = (\mathbb{Z}_p)^k$, $p > 2$: $h^* = H^*_G(-;\mathbb{Z}_p)$, $I = (c_1, \ldots, c_k)$,

(3) if $G = (\mathbb{S}^1)^k$: $h^* = H^*_G(-;\mathbb{Q})$, $I = H^*_G(\mathrm{pt};\mathbb{Q})$

Not difficult to repeat and get

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For $G = (\mathbb{Z}_2)^k$, or correspondingly $G = \mathbb{Z}_p$, $p > 2$, and $G = \mathbb{S}^1$ one can show that any G-invariant closed set

$\ell(Z) \leq \mathrm{coh.dim} Z + 1\,,$ or respectively $2\ell(Z) \leq \mathrm{coh.dim}\, Z + 1\,.$

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The case $G = (\mathbb{S}^1)^k$ can be reduced to $G = \mathbb{S}^1$. However it is not possible to compare $\ell(Z)$ and $\mathrm{coh.dim}\, Z$ for the group $G = (\mathbb{Z}_p)^k$, $p > 2$.

Theorem

Let V, W be two orth. representations of $G = \mathbb{Z}_p^k$ or $G = \mathbb{T}^k$ such that $V^G = W^G = \{0\}$. If $f : S(V) \xrightarrow{G} W$ is G-map, then

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Corollary

Let $G = \mathbb{Z}_p^k$ with $p > 2$, or $G = \mathbb{T}^k$ and V, W as above. Then for any $f : S(V) \xrightarrow{G} W$ $\dim_{\mathbb{R}} V > \dim_{\mathbb{R}} W$ implies $\mathrm{coh.dim} Z_f \geq 1\,.$

Indeed, $\dim_{\mathbb{R}} V - \dim_{\mathbb{R}} W - 1 = 2d(V) - 2d(W) - 1$ is integral, positive and odd. $\square$

We shall use a most general version of the Borsuk-Ulam theorem by Assadi in [2, page 23] (for p-torus) and Clapp and Puppe in [8, Theorem 6.4] for the torus and p-torus

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Theorem

Let G be a p-torus or a torus. Let X and Y be G-spaces with fixed-points-free actions; moreover, in the case of a torus action assume additionally that Y has finitely many orbit types. Suppose that $\tilde{H}_j(X) = \tilde{H}^j(X) = 0$ for $j < n$, Y is compact or paracompact and finite-dimensional, and $H_j(Y) = H^j(Y) = 0$ for $j \geq n$. Then there exists no G-equivariant map of X into Y.

Proof: Denote $m = \dim_{\mathbb{R}} V$ and $n = \dim_{\mathbb{R}} W$ and suppose

$$\mathrm{coh.dim} Z_f < m - n - 1.$$

Then,

$$\check{H}^i(Z_f) = 0, \text{ for any } i > m - n - 2.$$

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By using Poincaré-Alexander-Lefschetz duality and the long exact sequence of the pair
$(SV, SV \setminus Z_f)$, we conclude

$$0 = \check{H}^i(Z_f) = H_{m-1-i}(SV, SV \backslash Z_f) = \tilde{H}_{m-i-2}(SV \backslash Z_f), \text{ for } j = m-i-2$$

$$\tilde{H}_j(SV \setminus Z_f) = 0, \text{ for } j < n.$$

On the other hand, we have

$$H_j(W \setminus \{0\}) = H_j(SW) = 0, \text{ for } j \geq n.$$

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In particular, if $\dim_{\mathbb{R}} V > \dim_{\mathbb{R}} W$, for a G-map $f : S(V) \to W \setminus \{0\} \subset W$ it implies that $\mathrm{coh.dim} Z_f \geq 0$ and, consequently, $Z_f \neq \emptyset$, which gives a contradiction. □

Theorem

Let $G = (\mathbb{Z}_2)^k$, $(\mathbb{Z}_p)^k$ *or* $(\mathbb{S}^1)^k$, *with* $k \geq 1$.

- *Let* X *be a* G*-space and a* K*-orientable closed topological manifold such that* $\tilde{H}^i(X) = 0$ for $i < n - 1$.

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In the case $p = 0$, *i.e. the torus, suppose* Y *has finitely many orbit type.*

If $f : X \longrightarrow Y$ *is a* G*-map, then*

$$\dim f^{-1}(A) \geq n - m - 1.$$

Theorem (Characterization of p-toral groups, WM/DM/ES (12))

a) *Let G be a p-toral group $1 \hookrightarrow \mathbb{T}^k \to G \to P \to 1$. Then for the sphere $S(V)$ of a G-Hilbert space (orthogonal representation) V, $V^G = \{0\}$, $\dim V = \infty$, and finite dimensional orthogonal representation W of G such that $W^G = \{0\}$, and a G-map $f : S(V) \to W$ we have*

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b) *If G is not p-toral then $\exists$ an infinite-dim. fixed point free G-Hilbert space V, a finite dimensional representation W of G with $W^G = \{0\}$ and a G-map $f : S(V) \to W$ such that*

$$Z_f = \emptyset, \quad e.g. \;\; \dim Z_f = -1 < \infty\,.$$

An adaptation of proof of a B-U theorem of Bartsch, Clapp & D. Puppe, based on the Borel cohomology of stable cohomotopy theory and used the Segal conjecture (G. Carlson) $\widehat{A(G)} = \pi^0_{st}(BG)$.

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i) Equivariant cohomology theory $h^*_G(X) = \omega^0_{st}(X \times_G EG)$ the Borel construction for the stable cohomotopy theory;

ii) By the mentioned theorem $h^*_G(pt) = h^0_G(pt) = \omega^0_{st}(BG) = \pi^0_{st}(BG) = \widehat{A(G)}$, where the completion is taken with respect to the ideal $I := \ker \dim : A(G) \to \mathbb{Z}$.

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iii) Next, one should use the fact that the completion map $A(G) \to \widehat{A(G)}$ is injective if $\mathcal{P}$ is a finite p-group (E. Laitinen).

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On the other hand for any finite G there exists an element $\alpha \in A(G)$ such that $\alpha^n \neq 0$ for every $n \in \mathbb{N}$ (T. tom Dieck). Consequently, its image $\widehat{\alpha} \in \widehat{A(G)} = \omega^0_{st}(BG) = h^0_G(pt)$ has the same property for a p-group. This shows that $\ell(pt) = \infty$.

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From it follows that for every G-map $f : S(V) \to W$

$$\ell(Z_f) \geq \ell(S(V)) - \ell(S(W)) = \infty$$

as $\ell(S(W)) < \infty$. The remaining task is to adapt it for any toral p-group and show that also $\dim Z_f = \infty$. □

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In particular, if $d(W) < d(V)/p^{k-1}$, *then* $\dim Z_f \geq 0$,
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In particular if $G = \Gamma$ a finite group with an element of order p^2.

Tools

An index of type $i_{\mathbb{Z}_2}$ for a $\mathbb{Z}_{p^k}$ spaces but defined by use of the equivariant K_G^* theory.
Its definition, thus the value, depends on the orbits in X.

$$l_n(X) = (\mathcal{A}_{m,n}, K_G^*, R) - \textit{length index of} \;\; (X)\,. \tag{2}$$

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