Complex, symplectic and Kähler geometry

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Geometry

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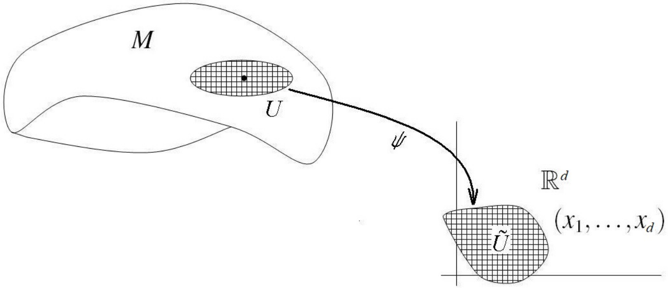

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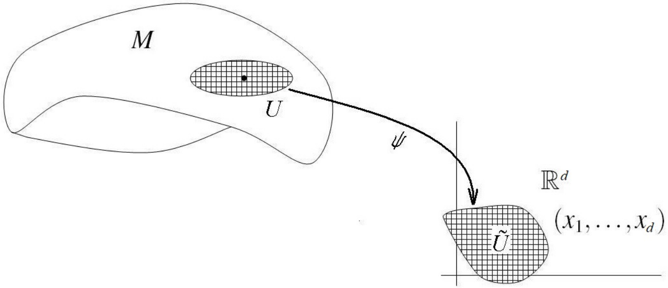

$\rightsquigarrow$ smooth functions on M, (tangent) vectors, etc.

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Main focus

Classify smooth (compact) manifolds with a given structure.

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This is an example of a number of topological obstructions for admitting a geometrical structure.

Topology $\rightsquigarrow$ Geometry.

Algebraic Geometry

Consider the ambient space $\mathbb{C}^n$.
Take $F_1, \ldots, F_m \in \mathbb{C}[z_1, \ldots, z_n]$.
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$S = V(F_1, \ldots, F_m)$, $F_i(z_0, \ldots, z_n)$ homogeneous polynomials, is a compact complex manifold (called *projective variety*).

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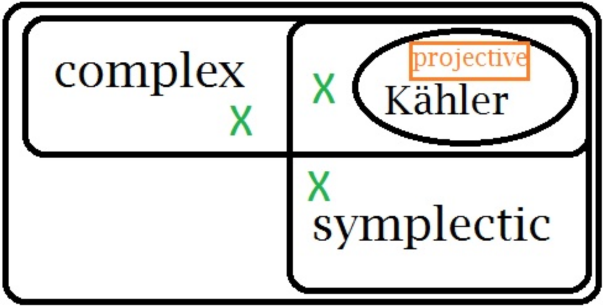

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Harmonic forms:
$\mathcal{H}^k(M) = \{\alpha \in \Omega^k(M) | \triangle\alpha = 0\} = \{\alpha | d\alpha = 0, d^*\alpha = 0\} \cong$
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$d\alpha = \partial\alpha + \bar{\partial}\alpha$
$\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)$,
$\bar{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$.

Dolbeault cohomology: $H^{p,q}(M) = \frac{\{\alpha \in \Omega^{p,q}(M) \,|\, \bar{\partial}\alpha = 0\}}{\{\alpha = \bar{\partial}\beta | \beta \in \Omega^{p,q-1}(M)\}}$.

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In particular, the Betti numbers satisfy:
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Analysis on manifolds $\rightsquigarrow$ Topology.

Kodaira-Thurston manifold

Kodaira, 1964

Complex manifold with $b_1 = 3$. It is given as

$$KT = \left\{ \begin{pmatrix} 1 & z & w \\ 0 & 1 & \bar{z} \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, (z,w) \in \mathbb{C}^2 \right\} / (\mathbb{Z} + \mathbb{Z}\, i)^2$$

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Let $KT = H \times S^1$, $\gamma = d\theta$. Symplectic form: $\omega = \alpha \wedge \gamma + \beta \wedge \eta$.

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Topological properties of Kähler manifolds

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Question

Does it exist a (compact) manifold M satisfying some topological property (e.g. b_{2k+1} even) admitting complex/symplectic structure but not admitting a Kähler structure?

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We extract an "invariant" from it:
Consider the equivalence relation $\sim$ between GCDAs generated by quasi-isomorphisms, $\psi : (A_1, d_1) \longrightarrow (A_2, d_2)$, i.e. morphisms inducing isomorphisms

$$\psi : H(A_1, d_1) \xrightarrow{\cong} H(A_2, d_2).$$

Then associate to $(\Omega X, d)$ its class in $(\text{GCDAs}/\sim)$.

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A minimal model $(\mathcal{M}_X, d)$ for X is a minimal model for $(\Omega X, d)$.

Minimal models

Theorem (Sullivan, 1977)

If either X is simply-connected or X is a nilpotent space, then the minimal model $(\mathcal{M}_X, d) \longrightarrow (\Omega X, d)$ codifies the rational homotopy of X. More specifically, $\mathcal{M}_X = \bigwedge V$, $V = \bigoplus_{n \geq 1} V^n$, where V^n is the vector space given by the degree n generators. Then

$$V^n \cong (\pi_n(X) \otimes \mathbb{R})^*\,,$$

and

$$H^n(\bigwedge V, d) = H^n(\Omega(X), d) = H^n(X).$$

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So the minimal model can be deduced *formally* from $H = H(A, d)$.
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A space X is formal if $(\Omega X, d)$ is formal.

Non-formal symplectic manifolds

The Kodaira-Thurston manifold is non-formal.

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Hence KT is non-formal.

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Nilmanifolds are never formal, unless they are tori.

Nilmanifolds

The Kodaira-Thurston manifold is a nilmanifold.

A nilmanifold is a quotient $M = G/\Gamma$, $d = \dim M$
G is a nilpotent group: $G_0 = G$, $G_i = [G, G_{i-1}]$, $i \geq 1$, and $G_d = 0$.
$\Gamma \subset G$ is a co-compact discrete subgroup.

Nilmanifolds are not simply-connected: $\pi_1(M) = \Gamma$.
The minimal model of M is $(\bigwedge(x_1, x_2, \dots, x_d), d)$ with $\deg x_i = 1$.
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Nilmanifolds are never formal, unless they are tori.

There are non-formal nilmanifolds which admit both complex and symplectic structures.
They cannot be Kähler.

Main results

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Theorem [Bazzoni-Fernández-Muñoz, 2014]

There is a simply-connected 6-dimensional manifold complex and symplectic which is not hard-Lefschetz. Hence it does not admit Kähler structures.

Non-formal 8-dimensional orbifold

Let $H = \left\{ A = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \,|\, a, b, c \in \mathbb{C} \right\}$

$\Gamma = \{A \in H \,|\, a, b, c \in \Lambda\}$

$\Lambda = \mathbb{Z} + \xi\mathbb{Z}$, $\xi = e^{2\pi i/3}$

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$\mathbb{C}/\Lambda \to H/\Gamma \to (\mathbb{C}/\Lambda) \times (\mathbb{C}/\Lambda)$, $(a, b, c) \mapsto (a, b)$.
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$M = H/\Gamma \times \mathbb{C}/\Lambda$ is 8-dimensional and non-formal but not simply-connected.

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Let $\mathbb{Z}_3$ act on M by $(a, b, c, z) \mapsto (\xi a, \xi b, \xi^2 c, \xi z)$,
so $(\alpha, \beta, \eta, \gamma) \mapsto (\xi\alpha, \xi\beta, \xi^2\eta, \xi\gamma)$.

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The CDGA of $\hat{M}$ is $(\bigwedge(\alpha, \bar{\alpha}, \beta, \bar{\beta}, \eta, \bar{\eta}, \gamma, \bar{\gamma})^{\mathbb{Z}_3}, d)$.
$\implies$ $\hat{M}$ is non-formal.

Symplectic resolution of singularities

Local model around a singular point: $B \cong (B(0,1)/\langle(\xi,\xi,\xi^2,\xi)\rangle, \hat{\omega})$.
With change of variables $(a',b',c',z') = (a, b - i\bar{c}, \bar{b} - ic, z)$,
$\hat{\omega} = -ida' \wedge d\bar{a}' - idb' \wedge d\bar{b}' - idc' \wedge d\bar{c}' - idz' \wedge d\bar{z}'$.
$\psi : B \xrightarrow{\cong} (B(0,1)/\langle(\xi,\xi,\xi^2,\xi)\rangle, \omega_{std})$.

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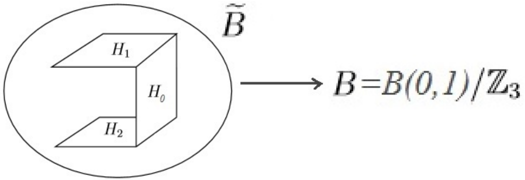

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Glue the symplectic forms on $\widehat{M}$ and $\widetilde{B}$ to get $(\widetilde{M}_s, \tilde{\omega})$.

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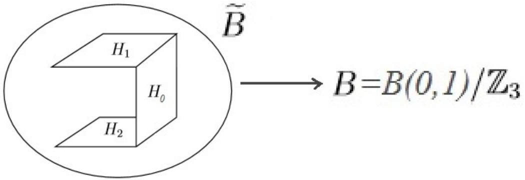

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QED