# $n$-GRAPHS, $n$-PERSON POSITIONAL GAMES, AND $\Delta$-CONJECTURE

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**Abstract:** An $n$-graph $\mathcal{G} = (V; E_1, \ldots, E_n)$ is defined as the complete graph whose edges are arbitrarily colored with $n$ colors.

An $n$-graph is called complementary connected (CC) if for every $i \in [n] = \{1, \ldots, n\}$ the complement $\overline{G}_i = (V, \overline{E}_i)$ of the $i$th chromatic component of $\mathcal{G}$ is connected on $V$.

$\Delta$ is a 3-colored triangle; $\Pi$ is a 2-graph whose both chromatic components are $P_4$.

$\Pi$ and $\Delta$ are inclusion-minimal CC.

Moreover, there are no others, but the trivial single-vertex graph. And still more, they are the only *locally* minimal, that is, each CC $n$-graph distinct from the above three contains a vertex such that after its deleting the remaining $n$-graph is still $CC$.

Based on these theorems, we construct a one-to-one correspondence between the $\Pi$- and $\Delta$-free $n$-graphs and $n$-person positional games with perfect information and characterize the normal forms of these games.

For $n = 2$ we obtain a characterization of the so-called read-once Boolean functions.

As another corollary, we obtain the so-called CIS (Cliques Intersect Stable sets) property of $\Pi$- and $\Delta$-free $n$-graphs:

$\cap_{i=1}^{n} S_i = \emptyset$, or equivalently, $|\cap_{i=1}^{n} S_i| = 1$
for any family of maximal stable sets $S_i \subseteq V$ of $G_i = (V, E_i)$, where $i \in [n] = \{1, \ldots, n\}$.

Characterizing the CIS $n$-graphs is a difficult problem already for $n = 2$. For example, every 2-graph is a subgraph of a CIS 2-graph. We give more partial results in this direction.

Yet, whether $\Delta$ might be a subgraph of a CIS $n$-graph?

**$\Delta$-conjecture (1978):** asserts that NO.

In other words, if an $n$-graph $\mathcal{G}$ contains a $\Delta$ then it cannot have the CIS property, that is, $\cap_{i \in [n]} S_i = \emptyset$ for some family of maximal stable sets $S_i \subseteq V$.

It would suffice to prove $\Delta$-conjecture for $n = 3$.

We also recall Tibor Gallai's (1967) decomposition of $\Delta$-free graphs into 2-graphs.

$\Delta$-conjecture, if true, would reduce characterizing CIS $n$-graphs to the case $n = 2$.

Related papers available online:

D. Andrade, E. Boros, and V. Gurvich, On graphs whose maximal cliques and stable sets intersect, RUTCOR Research Report RRR-17-2006, Rutgers University.

V. Gurvich, Decomposing complete edge-chromatic graphs and hypergraphs; Revisited, RUTCOR Research Report, RRR-29-2006, Rutgers University, Discrete Applied Math., 157 (2009) 3069–3085.

V. Gurvich, On exact blockers and anti-blockers, $\Delta$-conjecture, and related problems, RUTCOR Research Report, RRR-10-2010, Discrete Appl. Math. 159 (2011), 311-321.