Equistarable Bipartite Graphs

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Joint work with Endre Boros and Martin Milanič

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Outline

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1. Equistable graphs
2. Equistarable graphs
3. Special cases
   - Bipartite graphs
   - Forests

Equistable graphs

Payan, 1980

A **stable set** (or independent set) in a graph $G = (V, E)$ is a subset $S \subseteq V$ such that no two vertices in S are adjacent.

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Definition

A graph G is said to be **equistable** if there exists a mapping $\varphi : V \to [0,1]$ such that for all $S \subseteq V$,

$$S \text{ is a maximal stable set} \iff \varphi(S) := \sum_{v \in S} \varphi(v) = 1.$$

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Such a φ is called an **equistable weight function** of G.

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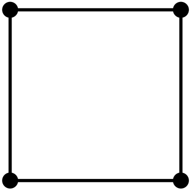

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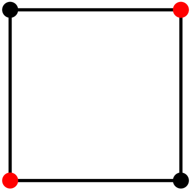

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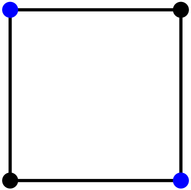

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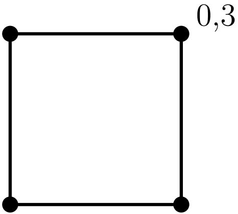

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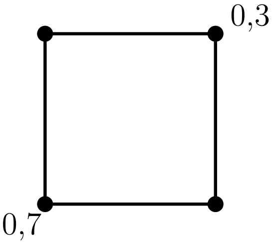

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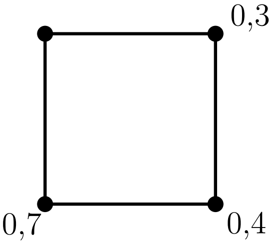

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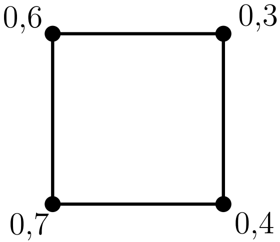

Equistable graphs
in connection to some other graph classes

Equistable

Equistable graphs
in connection to some other graph classes

Mahadev et al.

Strongly equistable

Equistable

Equistable graphs

in connection to some other graph classes

Strongly equistable

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Definition

Given a graph G, let $\mathscr{S}(G)$ be the set of all maximal stable sets of G, and $\mathscr{T}(G)$ the set of all other nonempty subsets of $V(G)$.

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Given a graph G, let $\mathscr{S}(G)$ be the set of all maximal stable sets of G, and $\mathscr{T}(G)$ the set of all other nonempty subsets of $V(G)$. A graph is said to be strongly equistable if for each $T \in \mathscr{T}(G)$ and for aech $\gamma \leq 1$,

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Equistable graphs
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Strongly equistable

$\Downarrow$

Equistable

Equistable graphs

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General partition

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Equistable

DeTemple et al.

Equistable graphs

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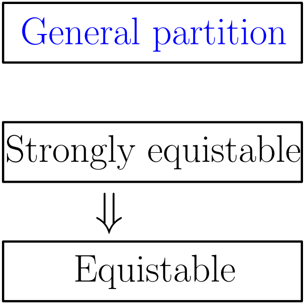

DeTemple et al.

Theorem (McAvaney et al., 1993)

A graph G is a general partition graph if and only if every edge of G is contained in a strong clique.

Equistable graphs
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General partition

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Strongly equistable

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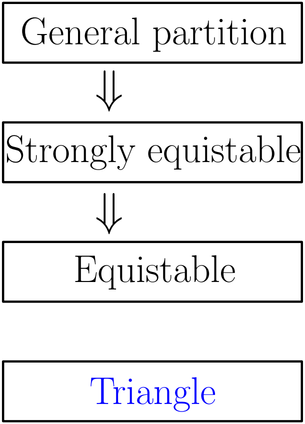

McAvaney et al.

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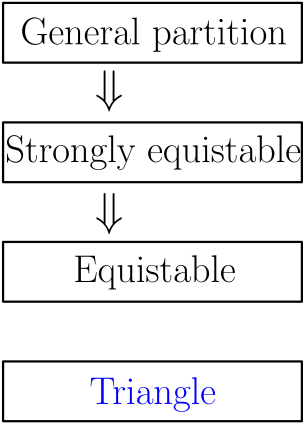

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Triangle condition

For every maximal stable set S in $G = (V, E)$ and every edge uv in $G - S$ there is a vertex $s \in S$ such that $\{u, v, s\}$ induces a triangle in G.

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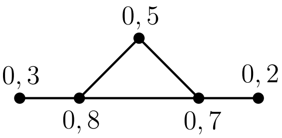

Equistable graphs
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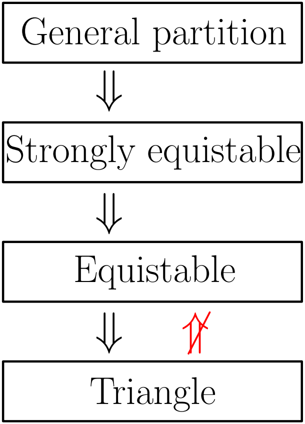

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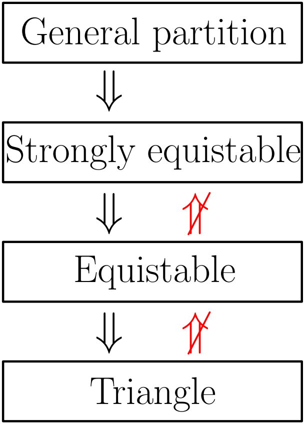

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General partition

$\Downarrow$ $\not\Uparrow$

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Milanič, Trotignon, 2014

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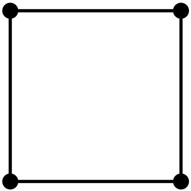

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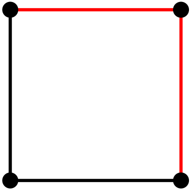

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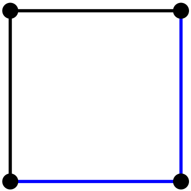

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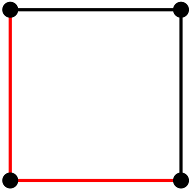

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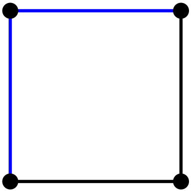

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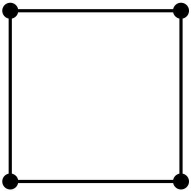

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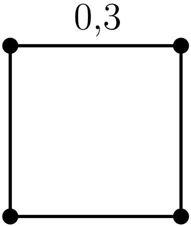

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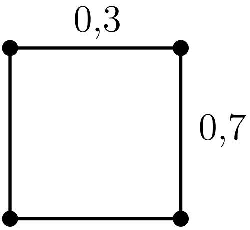

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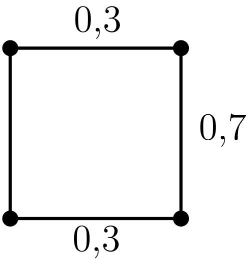

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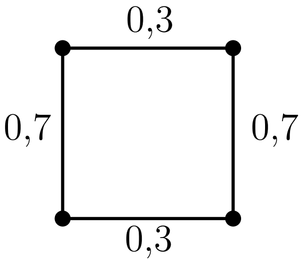
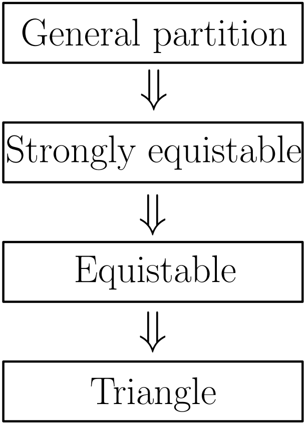


Equistarable graphs

in connection to some other graph classes



Equistarable graphs
in connection to some other graph classes

$\overline{L(G)}$

General partition

$\Downarrow$

Strongly equistable

$\Downarrow$

Equistable

$\Downarrow$

Triangle

Equistarable graphs

in connection to some other graph classes

G

$\overline{L(G)}$

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Equistarable graphs
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triangle-free graph G

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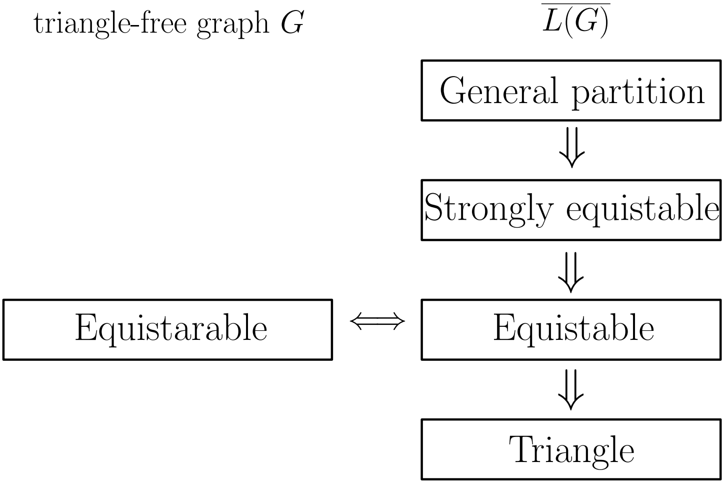

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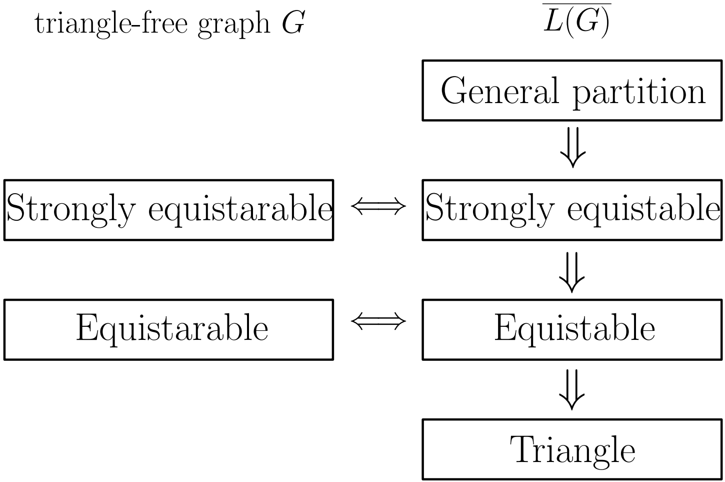

Equistarable graphs

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triangle-free graph G		$\overline{L(G)}$
$\forall$ component is a star or 2-internally extendable	$\Longleftrightarrow$	General partition
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Strongly equistarable	$\Longleftrightarrow$	Strongly equistable
		$\Downarrow$
Equistarable	$\Longleftrightarrow$	Equistable
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		Triangle

Recall: A graph G is 2-**extendable** if it is connected, contains a 2-matching and every 2-matching extends into a perfect matching.

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Definition

A graph is 2- **internally extendable** if every 2-matching can be extended to a perfect internal matching.

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P_5-constrained	$\Longleftrightarrow$	Triangle

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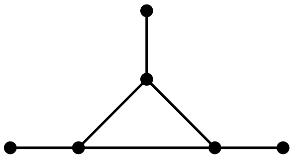

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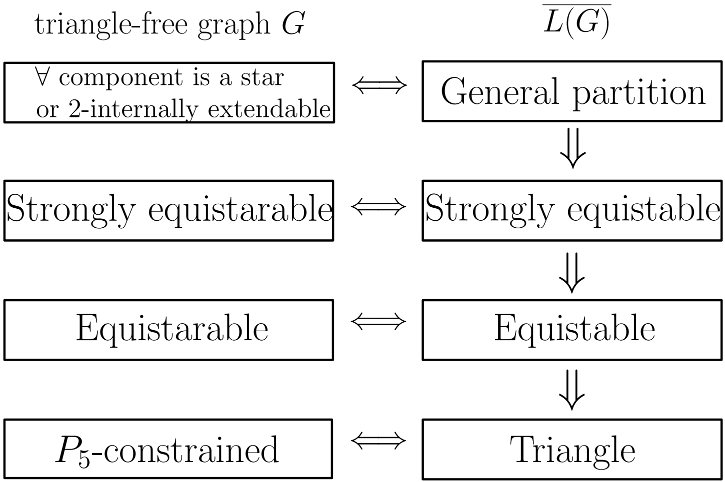
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Bipartite graphs

A graph is **bipartite** if its vertex set can be partitioned into two stable sets.

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Theorem

For a bipartite graph G the following are equivalent:
(a) Every connected component of G is either a star or 2-internally extendable.
(b) G is strongly equistarable.
(c) G is equistarable.

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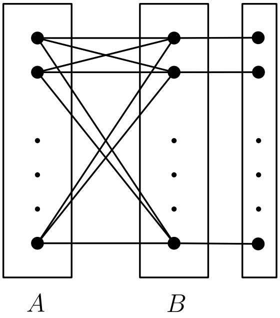

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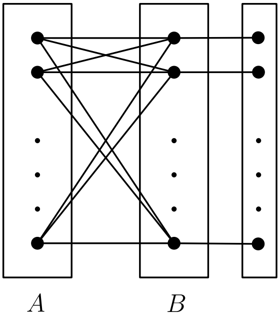

Suppose:
$|A| = k$
$|B| = l$

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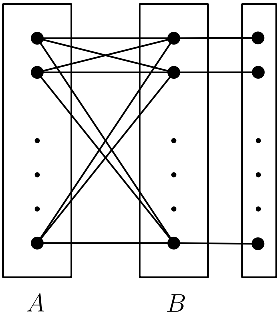

Suppose:
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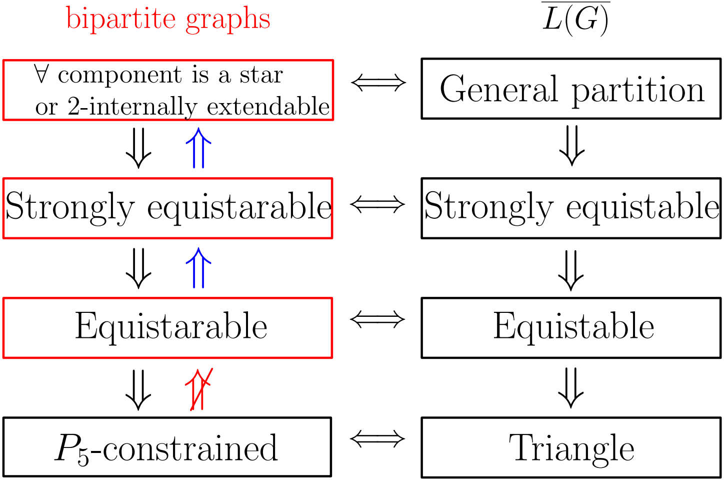
Every such graph with
$3 \leq l \leq k + 1$
is not 2-internally extendable.


bipartite graphs
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∀ component is a star or 2-internally extendable
General partition
Strongly equistarable
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P_5-constrained
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Forests

A **forest** is an acyclic graph.

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Theorem

For every forest F the following are equivalent:
(a) Every connected component of F either a star or 2-internally extendable.
(b) F is strongly equistarable.
(c) F is equistarable.
(d) F is P_5-constrained.

Proof sketch

Since forests are acyclic and therefore triangle-free,

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(d)⇒(a)

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Lemma

Every tree T with $|E(T)| \geq 1$ is almost 1-extendable.

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Lemma

Every tree T with $|E(T)| \geq 1$ is almost 1-extendable.

Let F be P_5-constrained. We can assume that F is connected and not a star.

Fix a 2-matching $M = \{e, f\}$, and consider the (unique) shortest path P between e and f.

We construct another matchig M' by putting in for every vertex of P, not covered by M, an arbitrary edge incident with it and not in P.

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Delete from the graph all the edges in $E(P)$.

What we have left is a forest F' consisting of some nontrivial trees, each of which contains at most one edge of $M' \cup M$. By the previous lemma matching $M' \cup M$ can be extended to a perfect internal matching of F.

We construct another matchig M' by putting in for every vertex of P, not covered by M, an arbitrary edge incident with it and not in P.
(Since F is P_5-constrained, all the vertices of P have degree ≥ 3.)

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Therefore, every connected component of F is either a star or 2-internally extendable.

Conclusion

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Open questions

- What is the complexity of recognizing equistarable graphs?
- Is every perfect equistable graph a general partition graph?

Thank you!