Characterization of Cyclic Schur Groups

Sergei Evdokimov

St.Petersburg Department of Steklov Mathematical Institute
Russia
(joint work with István Kovács and Ilia Ponomarenko)

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Outline

Definition

Let G be a group and $\mathbb{Q}G$ the group algebra of G over $\mathbb{Q}$.

An S-ring over G is a subalgebra $\mathcal{A}$ of $\mathbb{Q}G$ that contains 1 and is closed under the termwise multiplication and inversion.

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The partition $\mathcal{S}$ is stable: 1) $\{1\} \in \mathcal{S}$, 2) $X \in \mathcal{S} \Rightarrow X^{-1} \in \mathcal{S}$, 3) $X \cdot Y$ is a linear combination of $Z \in \mathcal{S}$ for all $X, Y \in \mathcal{S}$.

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Examples: 1) $\mathcal{A} = \mathbb{Q}G$ – trivial S-ring, 2) S-rings of rank 2, 3) tensor product $\mathcal{A}_1 \otimes \mathcal{A}_2$ is an S-ring over $G_1 \times G_2$.

Schur Theorem on Multipliers

For $X \subset G$ and $m \in \mathbb{Z}$ set $X^{(m)} = \{x^m : x \in X\}$.

Theorem Let $\mathcal{A}$ be an S-ring over an abelian group G. Then for any integer m coprime to |G| we have

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In other words the mapping $X \mapsto X^{(m)}, X \in \mathcal{S}$ is a bijection of $\mathcal{S}$.

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$$M = \{\sigma \in \mathrm{Aut}(G) : \exists \ m \in \mathbb{Z} \ \text{ such that } \ x^{\sigma} = x^m \ \forall x \in G\}.$$

Then $M = Z(\mathrm{Aut}(G))$ and the group M acts on the set $\mathcal{S}$.

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If G is cyclic, then $M = \mathrm{Aut}(G)$ and hence M acts transitively on the basic sets containing a generator of G.

Automorphism Group of an S-ring

Any basic $\mathcal{A}$-set X yields a Cayley graph $\mathcal{G}_X = (G, E_X)$ over G where $E_X = \{(g, xg) : g \in G, x \in X\}$. All of these graphs form a Cayley scheme $\mathcal{C} = (G, \{E_X\}_{X \in \mathcal{S}})$ over G.

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We have $\mathrm{Aut}(\mathcal{A}) = \mathrm{Aut}(\mathcal{A})_1 \cdot G_{right} \leq \mathrm{Sym}(G)$. Moreover, given $\sigma \in \mathrm{Aut}(\mathcal{A})_1$ we have $X^\sigma = X$ for all $X \in \mathcal{S}$.

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Examples: 1) $\mathrm{Aut}(\mathcal{A}) = G_{right}$ if $\mathcal{A} = \mathbb{Q}G$, 2) $\mathrm{Aut}(\mathcal{A}) = \mathrm{Sym}(G)$ if $\mathrm{rank}(\mathcal{A}) = 2$, 3) $\mathrm{Aut}(\mathcal{A}_1 \otimes \mathcal{A}_2) = \mathrm{Aut}(\mathcal{A}_1) \times \mathrm{Aut}(\mathcal{A}_2)$.

Schurian S-rings

For a permutation group $\Gamma \leq \mathrm{Sym}(G)$ with $G_{right} \leq \Gamma$ set

$$\mathcal{A}_\Gamma = \mathrm{span}\,\mathcal{S}_\Gamma, \quad \mathcal{S}_\Gamma = \mathrm{Orb}(\Gamma_1, G).$$

Then $\mathcal{A}_\Gamma$ is an S-ring over G. Such S-rings are called schurian.

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Special case: $\Gamma = K \cdot G_{right} \leq \mathrm{Hol}(G)$ where $K \leq \mathrm{Aut}(G)$. Then $\Gamma_1 = K$ and $\mathcal{S}_\Gamma = \mathrm{Orb}(K, G)$. The ring $\mathcal{A}_\Gamma$ is called cyclotomic and denoted by $\mathrm{Cyc}(K, G)$.

Generalized Wreath Product

Let U/L be an $\mathcal{A}$-section such that L is normal in G.

Definition We say that $\mathcal{A}$ is the U/L-wreath product if every basic set outside U is the union of L-cosets. The product is called proper if $L \neq 1$ and $U \neq G$.

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Theorem Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be S-rings over U and G/L. Suppose that U/L is both an $\mathcal{A}_1$- and $\mathcal{A}_2$-section and $(\mathcal{A}_1)_{U/L} = (\mathcal{A}_2)_{U/L}$. Then there is a uniquely determined S-ring $\mathcal{A}$ over G that is the U/L-wreath product such that $\mathcal{A}_U = \mathcal{A}_1$ and $\mathcal{A}_{G/L} = \mathcal{A}_2$.

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In this case we write $\mathcal{A} = \mathcal{A}_1 \wr_{U/L} \mathcal{A}_2 = \mathcal{A}_U \wr_{U/L} \mathcal{A}_{G/L}$.

Burnside – Schur Theorem (1933)

Theorem Every primitive permutation group containing a full cycle is either 2-transitive or isomorphic to a subgroup of the affine group $\mathrm{AGL}_1(p)$ where p is a prime.

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The theorem follows from the following statement in which we call an S-ring $\mathcal{A}$ over G normal if $\mathrm{Aut}(\mathcal{A}) \leq \mathrm{Hol}(G)$.

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Indeed, if Γ is a primitive group, then $\mathcal{A}_\Gamma$ is a primitive S-ring. If $\mathrm{rk}(\mathcal{A}_\Gamma) = 2$, then Γ is 2-transitive. Otherwise, $\mathcal{A}_\Gamma$ is normal. But then $\Gamma \leq \mathrm{Aut}(\mathcal{A}_\Gamma) \leq \mathrm{Hol}(\mathbb{Z}_p) = \mathrm{AGL}_1(p)$.

Decomposition Theorem (Leung – Man, EP)

Given an S-ring $\mathcal{A}$ over G set

$$\text{rad}(\mathcal{A}) = \text{rad}(X)$$

where X is a basic set of $\mathcal{A}$ that contains a generator of G and $\text{rad}(X) = \{g \in G : \ gX = X\}$.

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Theorem Let $\mathcal{A}$ be a circulant S-ring. Then

- if $\mathrm{rad}(\mathcal{A}) \neq 1$ then $\mathcal{A}$ is a proper generalized wreath product,
- if $\mathrm{rad}(\mathcal{A}) = 1$ then $\mathcal{A}$ is the tensor product of a normal cyclotomic S-ring and S-rings of rank 2.

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Corollary Any circulant S-ring with trivial radical is schurian.

Schurity of Generalized Wreath Product

Let us consider the U/L-wreath product

$$\mathcal{A} = \mathcal{A}_U \wr_{U/L} \mathcal{A}_{G/L}.$$

Theorem (EP) Suppose that $\mathcal{A}_{U/L}$ is the tensor product of a normal S-ring and S-rings of rank 2. Then $\mathcal{A}$ is schurian if and only if so are the S-rings $\mathcal{A}_U$ and $\mathcal{A}_{G/L}$.

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Corollary Suppose that $\mathrm{rad}(\mathcal{A}_{U/L}) = 1$. Then $\mathcal{A}$ is schurian if and only if so are the S-rings $\mathcal{A}_U$ and $\mathcal{A}_{G/L}$.

Definition and Properties

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Corollary A nilpotent (in particular, abelian) group the order of which is coprime to 6 is Schur only if it is cyclic.

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Corollary A nilpotent (in particular, abelian) group the order of which is coprime to 6 is Schur only if it is cyclic.

The smallest non-Schur group is $\mathbb{Z}_4 \times \mathbb{Z}_4$.

Previous Results

Cyclic Schur groups of order n:

- $n = p^k$ where p is an odd prime (Pöschel, 1974),
- $n = pq$ where p, q are primes (Klin – Pöschel, 1978),
- $n = 2^k$ (Kovács, 2009),
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- $n = pqr$ or $n = p^3q$ where p, q, r are primes (EP, 2010).

Let $\mathcal{A}$ be an S-ring over $\mathbb{Z}_{pqr}$. If $\mathrm{rad}(\mathcal{A}) = 1$, then $\mathcal{A}$ is schurian. Otherwise, $\mathcal{A} = \mathcal{A}_U \wr_{U/L} \mathcal{A}_{G/L}$ where $|U/L| = 1$ or p or q or r. So $\mathrm{rad}(\mathcal{A}_{U/L}) = 1$. Since $\mathcal{A}_U$ and $\mathcal{A}_{G/L}$ are schurian, so is $\mathcal{A}$.

Schur – Klin Hypothesis (1970's)

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The hypothesis was disproved in 2001 by EP.

Theorem Let $n = p_1 p_2 p_3 p_4$ where $p_i' s$ are odd primes such that $\{p_1, p_2\} \cap \{p_3, p_4\} = \emptyset$. Then a cyclic group of order n is not Schur whenever $\mathrm{GCD}(p_1 - 1, p_2 - 1, p_3 - 1, p_4 - 1) \geq 3$.

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To construct counterexamples the authors used generalized wreath product of S-rings. The smallest counterexample was on $n = 5^2 \cdot 13^2$ points.

Theorem

A cyclic group of order n is a Schur group if and only if n belongs to one of the following five (partially overlapped) families of integers:

$$p^k, \ p^k q, \ 2p^k q, \ pqr, \ 2pqr,$$

where p, q, r are distinct primes and $k \geq 0$ is an integer.

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Note. Cyclic groups of orders $4p^k$ and $4pq$ are also Schur.

The smallest non-Schur cyclic group has order 72.

Sketch of Proof

Given a positive integer m set

$$\Omega^*(m) = \begin{cases} \Omega(m), & \text{if } m \text{ is odd,} \\ \Omega(m/2), & \text{if } m \text{ is even.} \end{cases}$$

For example, $\Omega(4) = 2$ whereas $\Omega^*(4) = 1$.

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Theorem Let $n = n_1 n_2$ where n_1 and n_2 are coprime positive integers such that $\Omega^*(n_i) \geq 2$, $i = 1, 2$. Then a cyclic group of order n is not a Schur group.

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Note. An integer n satisfies the theorem condition if and only if it does not belong to any of the above five families of integers.

Sketch of Proof

For coprime $n_1 = ab$ and $n_2 = cd$ where $a, b, c, d \geq 3$ set

$$\mathcal{A}_1 = \mathrm{Cyc}(K_{ac}, \mathbb{Z}_{ac}), \quad \mathcal{A}_2 = \mathrm{Cyc}(K_{bc}, \mathbb{Z}_{bc}),$$

$$\mathcal{A}_3 = \mathrm{Cyc}(K_{ad}, \mathbb{Z}_{ad}), \quad \mathcal{A}_4 = \mathrm{Cyc}(K_b \times K_d, \mathbb{Z}_{bd})$$

with $K_m = \{1, -1\} \leq \mathbb{Z}_m^{\times}$ for all $m \geq 3$.

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$$\mathcal{A}_{1,2} = \mathcal{A}_1 \wr_c \mathcal{A}_2, \qquad \mathcal{A}_{3,4} = \mathcal{A}_3 \wr_d \mathcal{A}_4,$$

Then $(\mathcal{A}_{1,2})^{n_1} = \mathrm{Cyc}(K_a, \mathbb{Z}_a) \wr \mathrm{Cyc}(K_b, \mathbb{Z}_b) = (\mathcal{A}_{3,4})_{n_1}$.

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$$\mathcal{A} = \mathcal{A}_{1,2} \wr_{n_1} \mathcal{A}_{3,4}.$$

Theorem The S-ring $\mathcal{A}$ over $\mathbb{Z}_n$ is not schurian.

Theorem Let $\mathcal{A} = \mathcal{A}_U \wr_{U/L} \mathcal{A}_{G/L}$ be an S-ring over an abelian group G. Then

- $\mathrm{Aut}(\mathcal{A}) = \mathrm{Aut}(\mathcal{A})^U \wr_{U/L} \mathrm{Aut}(\mathcal{A})^{G/L}$,
- $\mathcal{A}$ is schurian if and only if there exist groups Δ_U and Δ_0 such that
  - $U_{right} \leq \Delta_U \leq \mathrm{Aut}(\mathcal{A}_U)$, $(G/L)_{right} \leq \Delta_0 \leq \mathrm{Aut}(\mathcal{A}_{G/L})$,
  - $\Delta_U \underset{2}{\approx} \mathrm{Aut}(\mathcal{A}_U)$, $\Delta_0 \underset{2}{\approx} \mathrm{Aut}(\mathcal{A}_{G/L})$,
  - $(\Delta_0)^{U/L} = (\Delta_U)^S$.

  Moreover, in this case $\mathrm{Aut}(\mathcal{A}) \underset{2}{\approx} \Delta_U \wr_{U/L} \Delta_0$.

Papers I

- S. Evdokimov, I. Kovács, I. Ponomarenko.
  Characterization of cyclic Schur groups.
  Preprint, 1–26, 2011.

- S. Evdokimov, I. Ponomarenko.
  Schurity of S-rings over a cyclic group and generalized wreath product of permutation groups.
  arXiv:1012.5393v2 [math.CO], 1–46, 2011 (accepted for publication in *St. Petersburg Math. J.*).