Half-arc-transitive graphs of particular orders

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May 16. 2012

Definition

Let X be a graph (without multiple edges, loops or semi-edges) with vertex set $V(X)$, edge set $E(X)$ and arc set $A(X)$. X is said to be vertex-transitive (VT), edge-transitive (ET) and arc-transitive (AT) if the automorphism group $Aut(X)$ is transitive on $V(X)$, $E(X)$ and $A(X)$ respectively.

Definition

A graph X is said to be half-arc-transitive (HAT) if it is VT and ET but not AT.

Proposition

A VT and ET graph X is a HAT graph iff for all $\{u, v\} \in E(X)$ there exists no $\alpha \in Aut(X)$ that interchanges u and v.

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Proposition (Marušič; 1998)

Let X be a graph having an automorphism ρ with two orbits U, V of length $n \geq 2$ such that $\{U, V\}$ is a bipartition of X. Then X is not a HAT graph.

Proposition

There is no HAT graph with less then 27 vertices.

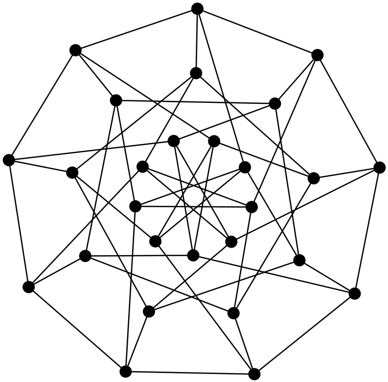

Figure: Doyle-Holt graph

Cayley graphs

Definition

Let G be a group and $S \subseteq G \setminus \{1\}$ inverse closed. *Cayley graph* $X = Cay(G; S)$ is a graph with vertex set $V(X) = G$ and $g \sim gs$ $\forall g \in G$, $s \in S$. Since S is inverse closed, X really is a graph.

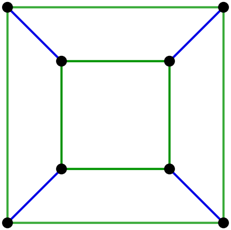

Figure: $Cay(D_8; \{\tau, \rho, \rho^{-1}\})$

Cayley graphs

Proposition (Feng, Wang, Zhou; 2007)

Let $X = Cay(G; S)$ be a HAT graph. Then there is no involution in S and no $\alpha \in Aut(G, S) = \{\alpha \in Aut(G);\ S^{\alpha} = S\}$ such that $s^{\alpha} = s^{-1}$ for any $s \in S$.

Proposition (Alspach, Marušič, Nowitz; 1994)

Every ET Cayley graph on an abelian group is also AT.

Metacirculants

Definition

Let $m \geq 1$ and $n \geq 2$ be integers. An automorphism of a graph is called *(m, n)-semiregular* if it has m orbits of size n and no other orbits on the vertices of the graph.

Definition

We call a graph X *(m, n)-metacirculant* if

1. there exists a (m, n)-semiregular automorphism ρ of X;
2. $\exists \sigma \in Aut(X) : \sigma^{-1}\rho\sigma = \rho^r$ for some $r \in \mathbb{Z}_n^*$ which cyclically permutes the orbits of ρ in such a way that there is a vertex of X fixed by σ^m and σ has one orbit on the set of orbits of ρ.

Metacirculants

Vertex set: $V(X) = \{x_i^j;\ i \in \mathbb{Z}_m,\ j \in \mathbb{Z}_n\}$

$x_i^j = (x_0^0)^{\sigma^i \rho^j}$, $0 \leq i \leq m-1 \Rightarrow (x_i^j)^\rho = x_i^{j+1}$ and $(x_i^j)^\sigma = x_{i+1}^{rj}$

Edge set: $S_i = \{s \in \mathbb{Z}_n \mid x_0^0 \sim x_i^s\}$, $0 \leq i \leq m-1$.

Every metacirculant is completely determined by the tuple $(m, n, r; S_0, \ldots, S_{m-1})$ which we call the *symbol* of X.

Metacirculants

Example

For each m, $n \geq 3$ and for each $r \in \mathbb{Z}_n^*$, where $r^m = \pm 1$, let $X(m, n; r)$ be a graph with $V(X) = \{x_i^j;\ i \in \mathbb{Z}_m,\ j \in \mathbb{Z}_n\}$ and $E(X) = \{\{x_i^j, x_{i+1}^{j \pm r^i}\};\ i \in \mathbb{Z}_m,\ j \in \mathbb{Z}_n\}$.
$X(m, n; r)$ is an (m, n)-metacirculant for $S_i = \emptyset\ \forall i \neq 1$ and $S_1 = \{1, -1\}$.

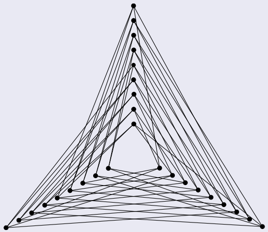

Figure: Doyle-Holt graph as a metacirculant $X(3, 9; 2)$

HAT graphs of particular orders

- p prime;
- np, p prime, $n > 1$ integer;
- p^n, p prime, $n > 1$ integer;
- $2p^2$, p prime.

HAT graphs of order prime p

Lemma (Alspach, Marušič, Nowitz; 1994)

There are no HAT graphs of order p.

Proof

Let X be a VT and ET graph of order p. Take g of order p in $Aut(X)$. Then $\langle g \rangle \leq Aut(X)$ is a regular subgroup and therefore X is a Cayley graph. Since all Cayley graphs of an abelian group are AT, there are no HAT graphs of prime order.

HAT graphs of order np, p prime, $n > 1$ integer

Classified:

1. $2p$, $3p$, $4p$;
2. pq, $2pq$, $q < p$ prime.

HAT graphs of order $2p$

Theorem (Cheng, Oxley; 1985)

All graphs that are VT and ET of order $2p$, where p is a prime are also AT.

Idea of proof

Find all graphs of order $2p$ that are VT and ET $\Rightarrow$ find automorphism groups of graphs $\Rightarrow$ chech that it acts transitively on the arcs in every case.

HAT graphs of order $3p$

Theorem (Alspach, Xu; 1994)

A graph of order $3p$ is a tetravalent HAT iff it is a metacirculant graph $X(3, p; r)$, where $p > 7$, $3|p-1$ and $r \in \mathbb{Z}_p^* \setminus \{\pm 1\}$. It is also a Cayley graph.

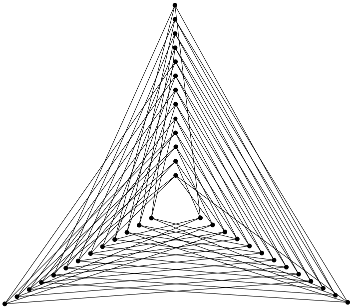

Figure: $X(3, 13; 3)$.

HAT graphs of order $4p$

Theorem (Feng, Wang, Zhou; 2006)

X is a tetravalent HAT graph of order $4p$ iff it is isomorphic to $X(4, p; r)$ where $r \in \mathbb{Z}_p^*$ and $r^4 = -1$. Note that such a graph exists for a given p iff $8 | p - 1$.

HAT graphs of order pq

Theorem (Wang; 1994)

Suppose that $5 \leq q < p$ are primes such that $q | p - 1$. Then X is a tetravalent HAT graph of order pq iff $X = X(q, p; r)$, with $(q, p) \neq (5, 11), (11, 23)$ and $r \in \mathbb{Z}_p^* \setminus \{\pm 1\}$.

HAT graphs of order $2pq$

Theorem (Feng, Kwak, Wang, Zhou; 2010)

Let p, q be odd primes, $q < p$ and X a tetravalent connected graph of order $2pq$. Then X is HAT iff $X \cong X(2q, p; r^k)$ where:

1. Cayley case: $(p, q) \neq (7, 3)$ and $q | p - 1$ and $X \cong Cay(G; \{cb^k, cb^{-k}, cb^k a, (cb^k a)^{-1}\})$ for $G = \langle a, b, c | \, a^p = b^q = c^2 = 1, a^b = a^r, ac = ca, bc = cb \rangle$, $k \in \mathbb{Z}_q^*$, $1 \leq k \leq \frac{q-1}{2}$ and $r \in \mathbb{Z}_p^*$ of order q or
2. Non-Cayley case: $4q | p - 1$ and k is odd, $1 \leq k \leq q - 1$, $r \in \mathbb{Z}_p^*$ of order $4q$.

HAT graphs of order $3p$, $4p$, pq, $2pq$

Idea of proof

1. $X(3, p; r)$, $X(4, p; r)$, $X(q, p; r)$ and $X(2q, p; r^k)$ (with mentioned conditions) are HAT.
2. The automorphism group of the corresponding HAT graph has a normal p-Sylow subgroup $\langle g \rangle$ of order p whose generator g is a semiregular automorphism. By analyzing the quotient graph with orbits of $\langle g \rangle$ as points and edges between these points whenever at least two points of corresponding orbits are connected in X, we can find automorphism $\sigma \in Aut(X)$ which cyclically permutes the orbits of $\langle g \rangle$ and satisfies all other conditions for being a metacirculant of described type.

HAT graphs of order p^n, p prime, $n > 1$ integer

Classified: p^2, p^3, p^4.

Idea of proof

Show that all graphs of order p^2, p^3, p^4 are Cayley graphs. Use that there is no HAT Cayley graph for an abelian group. Then look at the classification of groups for non-abelian groups of chosen order and for each of them check if there is a Cayley graph for one of them that is HAT.

HAT graphs of order p^2

Theorem (Marušič; 1985)

There are no HAT graphs of order p^2.

Proof

Let X be a VT and ET graph of order p^2. $Aut(X)$ has a p-Sylow subgroup P where P is transitive on $V(X)$ by Wielandt and $\forall g \in P : g^{p^2} = 1$.

1. If there is $g \in P$ of order p^2, $\langle g \rangle$ acts regularly on $V(X)$ and therefore X is a Cayley graph of a cyclic group. So X is AT.
2. $\forall g \in P \; g^p = 1$. Take $1 \neq h \in Z(P)$. $Aut(X)$ acts faithfully on $V(X)$ so h has at least one orbit of length p, say $v^{\langle h \rangle}$. Take $u \notin v^{\langle h \rangle}$. Since P is transitive on $V(X)$ there $\exists g \in P \; v^g = u$ and $\langle g, h \rangle$ is transitive on $V(X)$. Since $h \in Z(P)$ $\langle g, h \rangle$ is an abelian group of order p^2 and it's action on $V(X)$ is regular. So X is a Cayley graph for an abelian group and therefore AT.

HAT graphs of order p^3

Theorem (Xu; 1992):

X is a tetravalent HAT graph of order p^3, $p > 2$ iff it is isomorphic to a Cayley graph $Cay(G; \{b^i a, b^i a^{-1}, (b^i a)^{-1}, (b^i a^{-1})^{-1}\})$ for $1 \leq i \leq \frac{p-1}{2}$, where $G = \left\langle a, b \middle| a^{p^2} = b^p = 1, a^b = a^{1+p} \right\rangle$.
They are all metacirculants.

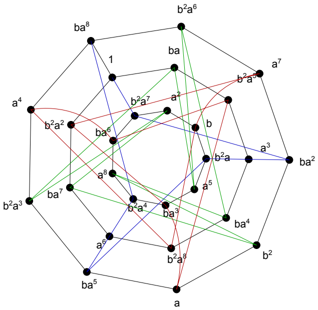

Figure: Doyle-Holt graph

HAT graphs of order p^4

It turns out that all tetravalent HAT graphs on p^4 vertices are metacirculant and are also Cayley graphs for groups:
$G_1(p) = \left\langle a,\ b \middle|\ a^{p^3} = b^p = 1,\ [a,\ b] = a^{p^2} \right\rangle$ and
$G_2(p) = \left\langle a,\ b \middle|\ a^{p^2} = b^{p^2} = 1,\ [a,\ b] = a^p \right\rangle$.

Theorem (Feng, Kwak, Xu, Zhou; 2007):

A connected tetravalent graph of order p^4, p odd prime, is HAT iff $p \geq 3$ and it is isomorphic to one of the Cayley graphs
$Cay(G_1(p), \{a,\ a^{-1}b^i,\ a^{-1},\ (a^{-1}b^i)^{-1}\})$ and
$Cay(G_2(p), \{b^i,\ ab^i,\ b^{-i},\ (ab^i)^{-1}\})$, $1 \leq i \leq (p-1)/2$.

HAT graphs of order $2p^2$

Theorem (Wang, Feng; 2010):

There are no tetravalent HAT graphs of order $2p^2$.

Idea of proof

Let X be a tetravalent HAT graph of order $2p^2$. As before $Aut(X)$ has a normal p-Sylow subgroup and it can be shown that X is a bipartite Cayley graph. With some calculations we see that every VT and ET Cayley graph of order $2p^2$ is also AT.

All tetravalent HAT graphs of particular orders

$\lvert V(X)\rvert$	all tetravalent HAT graphs
p	/
$2p$	/
$3p$	$X(3, p; r)$; $p \neq 7$, $3 \mid p-1$, $r \neq 1$, $r^3 = \pm 1$ (all Cayley)
$4p$	$X(4, p; r)$; $r^4 = -1$ and $8 \mid p-1$ (none Cayley)
qp	$X(q, p; r)$; $(q, p) \neq (5, 11), (11, 23)$, $q \mid p-1$, $r^q = \pm 1$
$2qp$	$X(2q, p; r^k)$ Cayley case: $(q, p) \neq (3, 7)$, $q \mid p-1$ non-Cayley case: $4q \mid p-1$, $1 \leq k \leq q-1$
p^2	/
p^3	$Cay(G; \{b^i a, b^i a^{-1}, (b^i a)^{-1}, (b^i a^{-1})^{-1}\})$
p^4	$Cay(G_1(p), \{a, a^{-1}b^i, a^{-1}, (a^{-1}b^i)^{-1}\})$ $Cay(G_2(p), \{b^i, ab^i, b^{-i}, (ab^i)^{-1}\})$, $1 \leq i \leq (p-1)/2$.
$2p^2$	/

Thank you!