Extremal 1-codes in distance-regular graphs of diameter 3

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Distance-regular graphs

- Let Γ be a graph of diameter d with vertex set $V\Gamma$, and $\Gamma_i(u)$ be the set of vertices of Γ at distance i from $u \in V\Gamma$.
- For $u, v \in V\Gamma$ with $\partial(u, v) = h$, denote

$$p^h_{ij}(u, v) := |\Gamma_i(u) \cap \Gamma_j(v)| .$$

- The graph Γ is *distance-regular* if the values of $p^h_{ij}(u, v)$ only depend on the choice of h, i, j and not on the particular vertices u, v.
- We call the numbers $p^h_{ij} := p^h_{ij}(u, v)$ $(0 \leq h, i, j \leq d)$ *intersection numbers*.

Distance-regular graphs

- Distance-regular graphs are regular with valency $k = p^0_{11}$.
- All intersection numbers can be determined from the *intersection array*

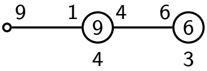
$$\{k, b_1, \dots, b_{d-1}; 1, c_2, \dots, c_d\} ,$$

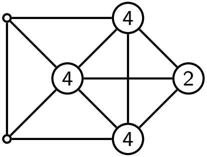
where $a_i := p^i_{1,i}$, $b_i := p^i_{1,i+1}$, $c_i := p^i_{1,i-1}$ and $a_i + b_i + c_i = k$ $(0 \leq i \leq d)$.
- Distance-regular graphs of diameter $d \leq 2$ are precisely the connected strongly regular graphs.
- **Problem**: Does a graph with a given intersection array exist? If so, is it unique? Can we determine all such graphs?

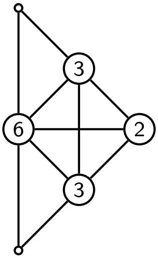
A small example

- Take the entries of the multiplication table of the Klein four-group as vertices.
- Two distinct vertices are adjacent if they are in the same row or column or if they share the value.
- The resulting graph is strongly regular and distance-regular with intersection array $\{9, 4; 1, 6\}$.

0	1	a	b
1	0	b	a
a	b	0	1
b	a	1	0







Distance-regular graphs of diameter 3

- When $d = 3$, the intersection array is

$$\{k, b_1, b_2; 1, c_2, c_3\}.$$

- Examples:
  - cycles C_6, C_7,
  - Hamming graphs $H(n, 3)$,
  - Johnson graphs $J(n, 3)$, $n \geq 6$,
  - generalized hexagons $GH(s, t)$,
  - odd graph on 7 points,
  - Sylvester graph,
  - and others.

Bose-Mesner algebra

- Let $A_0, A_1, \ldots A_d$ be binary matrices indexed by $V\Gamma$ with $(A_i)_{uv} = 1$ *iff* $\partial(u, v) = i$.
- These matrices can be diagonalized simultaneously and they share $d + 1$ eigenspaces.
- Let P be a $(d + 1) \times (d + 1)$ matrix with P_{ij} being the eigenvalue of A_j corresponding to the i-th eigenspace.
- Let Q be such that $PQ = |V\Gamma|I$.
- We call P the *eigenmatrix*, and Q the *dual eigenmatrix*.
- The matrices $\{A_0, A_1, \ldots A_d\}$ are the basis of the *Bose-Mesner algebra* $\mathcal{M}$, which has a second basis $\{E_0, E_1, \ldots E_d\}$ of minimal idempotents for each eigenspace.

Krein parameters

- In the Bose-Mesner algebra $\mathcal{M}$, the following relations are satisfied:

$$A_j = \sum_{i=0}^{d} P_{ij} E_i \quad \text{and} \quad E_j = \frac{1}{n} \sum_{i=0}^{d} Q_{ij} A_i \ .$$

- We also have

$$A_i A_j = \sum_{h=0}^{d} p^h_{ij} A_h \quad \text{and} \quad E_i \circ E_j = \frac{1}{n} \sum_{h=0}^{d} q^h_{ij} E_h \ ,$$

where $\circ$ is the entrywise matrix product.

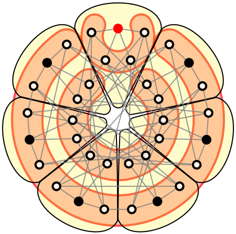
- The numbers q^h_{ij} are called the *Krein parameters* and are nonnegative algebraic real numbers.

Codes in distance-regular graphs

- An e-code C in a graph Γ is a set of vertices with $\partial(u, v) \geq 2e + 1$ for any distinct $u, v \in C$.
- The size of the code C in a distance-regular graph is limited by the sphere packing bound:

$$|C| \sum_{i=0}^{e} k_i \leq |V\Gamma|$$

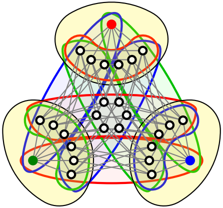
- If equality holds in the above bound, we call C a perfect e-code.



More bounds

- Let Γ be a distance-regular graph of diameter $d = 2e + 1$ and C an e-code in Γ.
- Then we have $|C| \leq p^d_{dd} + 2$.
  If equality holds, C is a *maximal* e-code.
- If a maximal code C exists, then $a_d p^d_{dd} \leq c_d$.
  If equality holds, C is a *locally regular* e-code.
- Another bound:

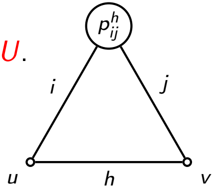
$$(|C| - 1) \sum_{i=0}^{e} p^d_{id} \leq k_d$$

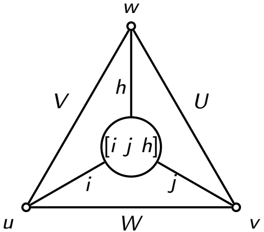


- If equality holds,
  C is a *last subconstituent perfect* e-code.

Triple intersection numbers

- In a distance regular graph, the intersection numbers $p^h_{ij} = |\Gamma_i(u) \cap \Gamma_j(v)|$ only depend on $h = \partial(u, v)$.
- Let $u, v, w \in V\Gamma$ with $\partial(u, v) = W$, $\partial(u, w) = V$ and $\partial(v, w) = U$.
- We define *triple intersection numbers* as
$$\begin{bmatrix} u & v & w \\ i & j & h \end{bmatrix} := |\Gamma_i(u) \cap \Gamma_j(v) \cap \Gamma_h(w)|$$
- $\begin{bmatrix} u & v & w \\ i & j & h \end{bmatrix}$ may depend on the particular choice of u, v, w!
- When u, v, w are fixed, we abbreviate $\begin{bmatrix} u & v & w \\ i & j & h \end{bmatrix}$ as $[i\ j\ h]$.





Codes and triple intersection numbers

- **Proposition**: Let Γ be a distance-regular graph of diameter $d = 2e + 1$ with a locally regular e-code C.
- Then, for u, v, w with $u \sim v$, $\partial(u, w) = d - 1$ and $v, w \in C$,

$$\begin{bmatrix} u & v & w \\ d & d & d \end{bmatrix} = 1$$

holds.

Infinite family 1

- We will study an infinite family of distance-regular graphs Γ with intersection array

$$\{(2r^2-1)(2r+1), 4r(r^2-1), 2r^2; 1, 2(r^2-1), r(4r^2-2)\}, \quad r > 1. \quad (1)$$

- Eigenvalues are

$$k = \theta_0 = (2r^2-1)(2r+1),\ \theta_1 = 2r^2+2r-1,\ \theta_2 = -1,\ \theta_3 = -2r^2+1.$$

- $\theta_2 = -1$ suggests that Γ might contain a perfect 1-code.

- The first two examples $r = 2, 3$:

$$\{35, 24, 8; 1, 6, 28\} \quad \text{and} \quad \{119, 96, 18; 1, 16, 102\}$$

appear in the list of feasible intersection arrays by Brouwer et al. [BCN89, pp. 425–431].

Infinite family 2

- Another infinite family we study is that of distance-regular graphs Γ with intersection array

$$\{2r^2(2r+1), (2r-1)(2r^2+r+1), 2r^2; 1, 2r^2, r(4r^2-1)\}, \quad r \geq 1. \qquad (2)$$

- Eigenvalues are

$$k = \theta_0 = 2r^2(2r+1), \quad \theta_1 = r(2r+1), \quad \theta_2 = 0, \quad \theta_3 = -r(2r+1).$$

- Since $\theta_1 = a_3$, these graphs are Shilla graphs [KP10].
- For $r = 1$ we have the Hamming graph $H(3,3)$.
- The next example $r = 2$:

$$\{40, 33, 8; 1, 8, 30\}$$

  appears in the list of feasible intersection arrays by Brouwer et al. [BCN89, pp. 425–431].

Common properties

- Let Γ be a graph with intersection array (1) or (2).
- Then Γ has diameter 3 and is formally self-dual.
- The Krein parameters $q^3_{11}, q^1_{13}, q^1_{31}$ of Γ vanish.
- **Lemma**: Let u, v be vertices of Γ with $\partial(u, v) = 3$. Then there exists a unique locally regular 1-code C such that $u, v \in C$.
- **Theorem**: For $r > 1$, Γ does not exist.

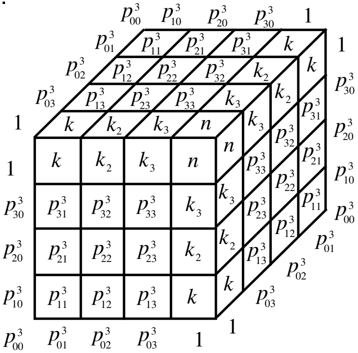
Computing triple intersection numbers

- We have $3d^2$ equations connecting triple intersection numbers to p_{ij}^h:

$$\sum_{\ell=1}^{d} [\ell\ j\ h] = p_{jh}^U - [0\ j\ h],$$

$$\sum_{\ell=1}^{d} [i\ \ell\ h] = p_{ih}^V - [i\ 0\ h],$$

$$\sum_{\ell=1}^{d} [i\ j\ \ell] = p_{ij}^W - [i\ j\ 0].$$



- All triple intersection numbers are nonnegative integers.

Computing triple intersection numbers (2)

[1 1 1] = 0 Δ	[1 2 1] = 0 Δ	[1 3 1] = 0 Δ
[1 1 2] = 0 Δ	[1 2 2] = α	[1 3 2] =
[1 1 3] = 0 Δ	[1 2 3] =	[1 3 3] =
[2 1 1] = 0 Δ	[2 2 1] = γ	[2 3 1] =
[2 1 2] = β	[2 2 2] =	[2 3 2] =
[2 1 3] =	[2 2 3] =	[2 3 3] =
[3 1 1] = 0 Δ	[3 2 1] =	[3 3 1] =
[3 1 2] =	[3 2 2] =	[3 3 2] =
[3 1 3] =	[3 2 3] =	[3 3 3] = δ

Computing triple intersection numbers (2)

$[1\ 1\ 1] = 0$ Δ	$[1\ 2\ 1] = 0$ Δ	$[1\ 3\ 1] = 0$ Δ
$[1\ 1\ 2] = 0$ Δ	$[1\ 2\ 2] = \alpha$	$[1\ 3\ 2] = c_3 - \alpha$
$[1\ 1\ 3] = 0$ Δ	$[1\ 2\ 3] = c_3 - \alpha$	$[1\ 3\ 3] =$
$[2\ 1\ 1] = 0$ Δ	$[2\ 2\ 1] = \gamma$	$[2\ 3\ 1] = c_3 - \gamma$
$[2\ 1\ 2] = \beta$	$[2\ 2\ 2] =$	$[2\ 3\ 2] =$
$[2\ 1\ 3] = c_3 - \beta$	$[2\ 2\ 3] =$	$[2\ 3\ 3] =$
$[3\ 1\ 1] = 0$ Δ	$[3\ 2\ 1] = c_3 - \gamma$	$[3\ 3\ 1] =$
$[3\ 1\ 2] = c_3 - \beta$	$[3\ 2\ 2] =$	$[3\ 3\ 2] =$
$[3\ 1\ 3] =$	$[3\ 2\ 3] =$	$[3\ 3\ 3] = \delta$

Computing triple intersection numbers (2)

$[1\ 1\ 1] = 0$ Δ	$[1\ 2\ 1] = 0$ Δ	$[1\ 3\ 1] = 0$ Δ
$[1\ 1\ 2] = 0$ Δ	$[1\ 2\ 2] =$ α	$[1\ 3\ 2] =$ $c_3 - \alpha$
$[1\ 1\ 3] = 0$ Δ	$[1\ 2\ 3] =$ $c_3 - \alpha$	$[1\ 3\ 3] =$ $a_3 - c_3 + \alpha$
$[2\ 1\ 1] = 0$ Δ	$[2\ 2\ 1] =$ γ	$[2\ 3\ 1] =$ $c_3 - \gamma$
$[2\ 1\ 2] =$ β	$[2\ 2\ 2] =$	$[2\ 3\ 2] =$
$[2\ 1\ 3] =$ $c_3 - \beta$	$[2\ 2\ 3] =$	$[2\ 3\ 3] =$
$[3\ 1\ 1] = 0$ Δ	$[3\ 2\ 1] =$ $c_3 - \gamma$	$[3\ 3\ 1] =$ $a_3 - c_3 + \gamma$
$[3\ 1\ 2] =$ $c_3 - \beta$	$[3\ 2\ 2] =$	$[3\ 3\ 2] =$
$[3\ 1\ 3] =$ $a_3 - c_3 + \beta$	$[3\ 2\ 3] =$	$[3\ 3\ 3] =$ δ

Computing triple intersection numbers (2)

$[1\ 1\ 1] = 0$ Δ	$[1\ 2\ 1] = 0$ Δ	$[1\ 3\ 1] = 0$ Δ
$[1\ 1\ 2] = 0$ Δ	$[1\ 2\ 2] = \alpha$	$[1\ 3\ 2] = c_3 - \alpha$
$[1\ 1\ 3] = 0$ Δ	$[1\ 2\ 3] = c_3 - \alpha$	$[1\ 3\ 3] = a_3 - c_3 + \alpha$
$[2\ 1\ 1] = 0$ Δ	$[2\ 2\ 1] = \gamma$	$[2\ 3\ 1] = c_3 - \gamma$
$[2\ 1\ 2] = \beta$	$[2\ 2\ 2] =$	$[2\ 3\ 2] =$
$[2\ 1\ 3] = c_3 - \beta$	$[2\ 2\ 3] =$	$[2\ 3\ 3] = p^3_{33} + c_3 - a_3 - 1 - \alpha - \delta$
$[3\ 1\ 1] = 0$ Δ	$[3\ 2\ 1] = c_3 - \gamma$	$[3\ 3\ 1] = a_3 - c_3 + \gamma$
$[3\ 1\ 2] = c_3 - \beta$	$[3\ 2\ 2] =$	$[3\ 3\ 2] = p^3_{33} + c_3 - a_3 - 1 - \gamma - \delta$
$[3\ 1\ 3] = a_3 - c_3 + \beta$	$[3\ 2\ 3] = p^3_{33} + c_3 - a_3 - 1 - \beta - \delta$	$[3\ 3\ 3] = \delta$

Computing triple intersection numbers (2)

$[1\ 1\ 1] = 0$ Δ	$[1\ 2\ 1] = 0$ Δ	$[1\ 3\ 1] = 0$ Δ
$[1\ 1\ 2] = 0$ Δ	$[1\ 2\ 2] =$ α	$[1\ 3\ 2] =$ $c_3 - \alpha$
$[1\ 1\ 3] = 0$ Δ	$[1\ 2\ 3] =$ $c_3 - \alpha$	$[1\ 3\ 3] =$ $a_3 - c_3 + \alpha$
$[2\ 1\ 1] = 0$ Δ	$[2\ 2\ 1] =$ γ	$[2\ 3\ 1] =$ $c_3 - \gamma$
$[2\ 1\ 2] =$ β	$[2\ 2\ 2] =$	$[2\ 3\ 2] =$ $p^3_{23} - c_3 + \gamma - [2\ 3\ 3]$
$[2\ 1\ 3] =$ $c_3 - \beta$	$[2\ 2\ 3] =$ $p^3_{23} - c_3 + \alpha - [3\ 2\ 3]$	$[2\ 3\ 3] =$ $p^3_{33} + c_3 - a_3 - 1 - \alpha - \delta$
$[3\ 1\ 1] = 0$ Δ	$[3\ 2\ 1] =$ $c_3 - \gamma$	$[3\ 3\ 1] =$ $a_3 - c_3 + \gamma$
$[3\ 1\ 2] =$ $c_3 - \beta$	$[3\ 2\ 2] =$ $p^3_{32} - c_3 + \beta - [3\ 3\ 2]$	$[3\ 3\ 2] =$ $p^3_{33} + c_3 - a_3 - 1 - \gamma - \delta$
$[3\ 1\ 3] =$ $a_3 - c_3 + \beta$	$[3\ 2\ 3] =$ $p^3_{33} + c_3 - a_3 - 1 - \beta - \delta$	$[3\ 3\ 3] =$ δ

Computing triple intersection numbers (2)

$[1\ 1\ 1] = 0$ Δ	$[1\ 2\ 1] = 0$ Δ	$[1\ 3\ 1] = 0$ Δ
$[1\ 1\ 2] = 0$ Δ	$[1\ 2\ 2] =$ α	$[1\ 3\ 2] =$ $c_3 - \alpha$
$[1\ 1\ 3] = 0$ Δ	$[1\ 2\ 3] =$ $c_3 - \alpha$	$[1\ 3\ 3] =$ $a_3 - c_3 + \alpha$
$[2\ 1\ 1] = 0$ Δ	$[2\ 2\ 1] =$ γ	$[2\ 3\ 1] =$ $c_3 - \gamma$
$[2\ 1\ 2] =$ β	$[2\ 2\ 2] =$ $p^3_{22} - \gamma - [2\ 2\ 3]$	$[2\ 3\ 2] =$ $p^3_{23} - c_3 + \gamma - [2\ 3\ 3]$
$[2\ 1\ 3] =$ $c_3 - \beta$	$[2\ 2\ 3] =$ $p^3_{23} - c_3 + \alpha - [3\ 2\ 3]$	$[2\ 3\ 3] =$ $p^3_{33} + c_3 - a_3 - 1 - \alpha - \delta$
$[3\ 1\ 1] = 0$ Δ	$[3\ 2\ 1] =$ $c_3 - \gamma$	$[3\ 3\ 1] =$ $a_3 - c_3 + \gamma$
$[3\ 1\ 2] =$ $c_3 - \beta$	$[3\ 2\ 2] =$ $p^3_{32} - c_3 + \beta - [3\ 3\ 2]$	$[3\ 3\ 2] =$ $p^3_{33} + c_3 - a_3 - 1 - \gamma - \delta$
$[3\ 1\ 3] =$ $a_3 - c_3 + \beta$	$[3\ 2\ 3] =$ $p^3_{33} + c_3 - a_3 - 1 - \beta - \delta$	$[3\ 3\ 3] =$ δ

Krein condition

- **Theorem** ([BCN89, Theorem 2.3.2], [CJ08, Theorem 3]):
  Let Γ be a distance-regular graph of diameter d,
  Q its dual eigenmatrix, and q_{ij}^h its Krein parameters.

- $q_{ij}^h = 0$ iff for all triples $u, v, w \in V\Gamma$:

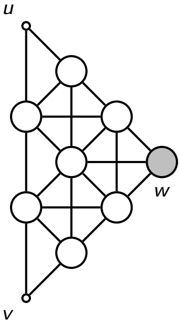
$$\sum_{r,s,t=0}^{d} Q_{ri} Q_{sj} Q_{th} \begin{bmatrix} u & v & w \\ r & s & t \end{bmatrix} = 0$$

- This gives a new equation
  in terms of triple intersection numbers.

The case $U = V = W = 3$

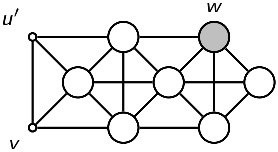
- Let Γ be a distance-regular graph with intersection array (1) or (2).
- If we choose $u, v, w \in V\Gamma$ such that $\partial(u, v) = \partial(u, w) = \partial(v, w) = 3$, we obtain a single solution with

$$\begin{bmatrix} u & v & w \\ 1 & 3 & 3 \end{bmatrix} = \begin{bmatrix} u & v & w \\ 3 & 1 & 3 \end{bmatrix} = \begin{bmatrix} u & v & w \\ 3 & 3 & 1 \end{bmatrix} = 0,$$
$$\begin{bmatrix} u & v & w \\ 2 & 3 & 3 \end{bmatrix} = \begin{bmatrix} u & v & w \\ 3 & 2 & 3 \end{bmatrix} = \begin{bmatrix} u & v & w \\ 3 & 3 & 2 \end{bmatrix} = 0,$$
$$\begin{bmatrix} u & v & w \\ 3 & 3 & 3 \end{bmatrix} = p^3_{33} - 1.$$



- As $c_3 = a_3 p^3_{33}$, there is a locally regular 1-code C in Γ with $u, v, w \in C$.

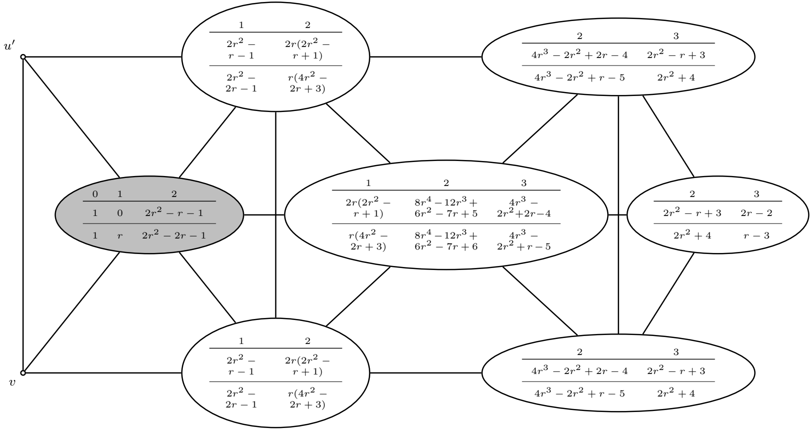
The case $\{U, V, W\} = \{1, 2, 3\}$



- Let C be a locally regular 1-code in Γ containing vertices v and w.
- For any $u' \in V\Gamma$ with $u' \sim v$ and $\partial(u', w) = 2$ we have $\begin{bmatrix} u' & v & w \\ 3 & 3 & 3 \end{bmatrix} = 1$.
- If Γ has intersection array (1), then there is no solution and Γ does not exist.
- If Γ has intersection array (2), then there is a single solution with $\begin{bmatrix} u' & v & w \\ 1 & 1 & 3 \end{bmatrix} = r$.

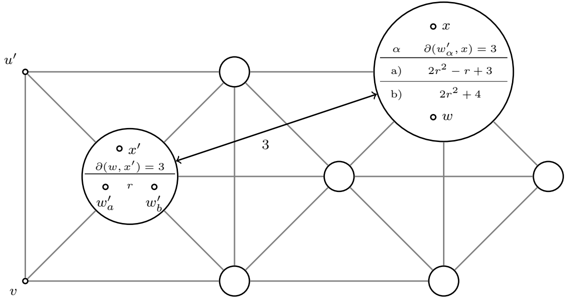
The case $U = V = W = 1$

- Let Γ be a distance-regular graph with intersection array (2).
- We obtain two solutions:



Counting solutions

- Let t and $a_1 - t$ be the numbers of vertices w'_a and w'_b such that $\begin{bmatrix} u' & v & w'_a \\ 2 & 3 & 3 \end{bmatrix} = 2r^2 - r + 3$ and $\begin{bmatrix} u' & v & w'_b \\ 2 & 3 & 3 \end{bmatrix} = 2r^2 + 4$.



- By comparing counts of pairs (w, x') and (w'_α, x), $\alpha \in \{a, b\}$ of vertices at distance 3, we obtain $t = \dfrac{r(2r-1)(3-r)}{r+1}$.

Ruling out family 2

- Case $r = 2$: we have $a_1 - t = 4$ vertices w'_b, but $\begin{bmatrix} u' & v & w'_b \\ 3 & 3 & 3 \end{bmatrix} = r - 3 < 0$, so the graph does not exist.
- Case $r = 3$: as $a_1 = 15$ and $t = 0$, for all neighbours w' of u' and v we have $\begin{bmatrix} u' & v & w' \\ 1 & 1 & 1 \end{bmatrix} = r = 3$, so $\Lambda(u', v)$ does not exist.
- Case $r > 3$: $t < 0$, contradiction.

Families with codes

- **Proposition**: Let Γ be a distance-regular graph of diameter 3 with a 1-code C that is locally regular and last subconstituent perfect.

- Set $a := a_3$, $p := p^3_{33}$ and $c := c_2$.
Then Γ has intersection array

  a) $\{a(p+1), cp, a+1; 1, c, ap\}$, or

  b) $\{a(p+1), (a+1)p, c; 1, c, ap\}$.

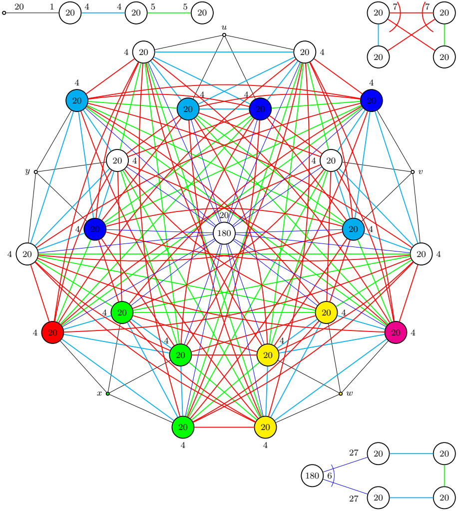
- **Conjecture**: A distance regular graph with intersection array a) is a subgraph of a Moore graph or has $a = c + 1$.

Examples

intersection array	status	intersection array	status
$\{5,4,2;1,1,4\}$	! Sylvester	$\{6,4,2;1,2,3\}$	! $H(3,3)$
$\{35,24,8;1,6,28\}$	$\nexists$	$\{12,10,2;1,2,8\}$	?
$\{44,30,5;1,3,40\}$	$\nexists$ [KP10]	$\{12,10,3;1,3,8\}$	! Doro
$\{48,35,9;1,7,40\}$	?	$\{18,10,4;1,4,9\}$	! $J(9,3)$
$\{49,36,8;1,6,42\}$	?	$\{24,21,3;1,3,18\}$	?
$\{54,40,7;1,5,48\}$	?	$\{25,24,3;1,3,20\}$	?
$\{55,54,2;1,1,54\}$	? in Moore(57)	$\{30,28,2;1,2,24\}$	?
$\{63,48,10;1,8,54\}$	?	$\{40,33,3;1,3,30\}$	?
$\{80,63,11;1,9,70\}$	?	$\{40,33,8;1,8,30\}$	$\nexists$
$\{99,80,12;1,10,88\}$	?	$\{50,44,5;1,5,40\}$	?
$\{119,96,18;1,16,102\}$	$\nexists$	$\{60,52,10;1,10,48\}$	?
		$\{65,56,5;1,5,52\}$	?
		$\{72,70,8;1,8,63\}$	?
		$\{75,64,8;1,8,60\}$	?
		$\{80,63,12;1,12,60\}$	?

An open case: $\{80, 63, 12; 1, 12, 60\}$

- We have much information about the structure.
- No costruction or proof of nonexistence is known.
- The third subconstituent is antipodal with intersection array $\{20, 15, 1; 1, 5, 20\}$ – also an open case.



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