Cubic Cayley graphs and snarks

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UP FAMNIT, Feb 2012

Outline

I. Snarks

II. Independent sets in cubic graphs

III. Non-existence of $(2, s, 3)$-Cayley snarks

IV. Snarks and $(2, s, t)$-Cayley graphs

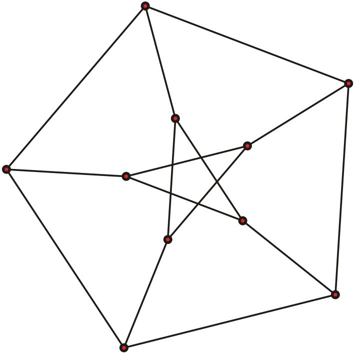
I. Snarks

Snarks

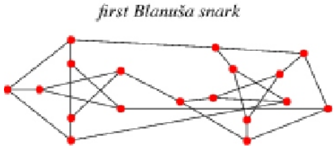
A snark is a connected, bridgeless cubic graph with chromatic index equal to 4.

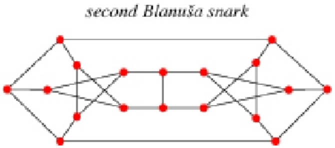
non-snark = bridgeless cubic 3-edge colorable graph

The Petersen graph is a snark



Blanuša Snarks (1946)

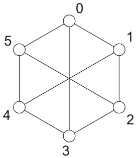




Cayley snarks?

A Cayley graph $Cay(G,S)$ on a group G relative to a subset $S = S^{-1} \subseteq G \setminus \{1\}$ has vertex set G and edges of the form $\{g, gs\}$, $g \in G$, $s \in S$.

Example: $Cay(\mathbb{Z}_6, \{\pm 1, 3\})$.



Are there snarks amongst Cayley graphs? (Alspach, Liu and Zhang, 1996)

Snarks

Nedela, Škoviera, Combin., 2001

If there exists a Cayley snark, then there is a Cayley snark $Cay(G, \{a, x, x^{-1}\})$ where x has odd order, $a^2 = 1$, and $G = \langle a, x \rangle$ is either a non-abelian simple group, or G has a unique non-trivial proper normal subgroup H which is either simple non-abelian or the direct product of two isomorphic non-abelian simple groups, and $|G : H| = 2$.

Potočnik, JCTB, 2004

The Petersen graph is the only vertex-transitive snark containing a solvable transitive subgroup of automorphisms.

Snarks

The hunting for vertex-transitive/Cayley snarks is essentially a special case of the Lovász question (1969) regarding hamiltonian paths/cycles.

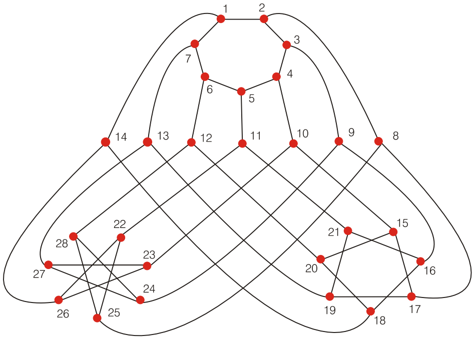
Existence of a hamiltonian cycle implies that the graph is 3-edge colorable, and thus a non-snark.

Hamiltonicity problem is hard, the snark problem is hard too, but should be easier to deal with.

The Coxeter graph is not a snark

vs

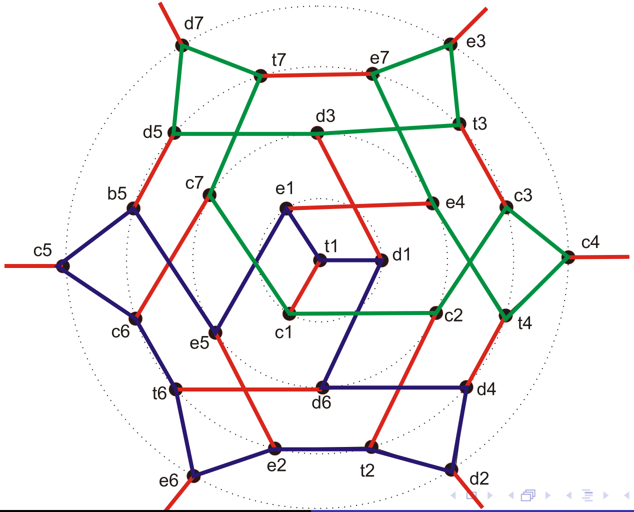
the Coxeter graph is not hamiltonian



The Coxeter graph is not a snark (easy)

vs

the Coxeter graph is not hamiltonian (harder)



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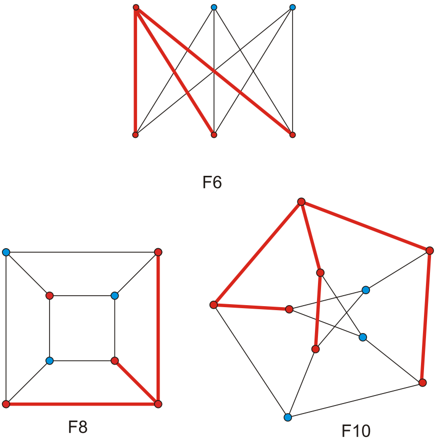
- Type 3: $S = \{a, x, x^{-1}\}$, where $a^2 = 1$ and x is of odd order.
  The general case still open. We will give an argument in the special case when the order of ax is 3 (smallest nontrivial case).

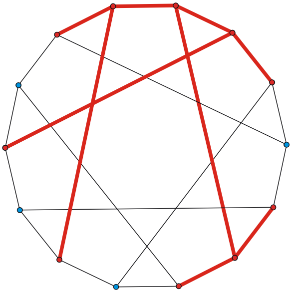
II. Independent sets in cubic graphs

Restriction to cyclically 4-edge-connected cubic graphs.

Given a connected graph X, a subset $F \subseteq E(X)$ is called cycle-separating if $X - F$ is disconnected and at least two of its components contain cycles. We say that X is cyclically k-edge-connected if no set of fewer than k edges is cycle-separating in X.


F6
F8
F10

Independent sets and their complements in cubic graphs



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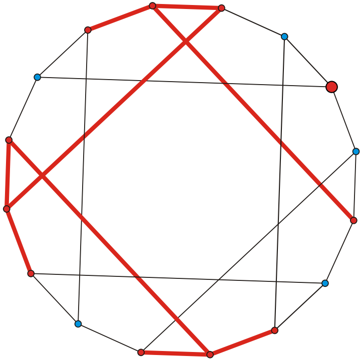
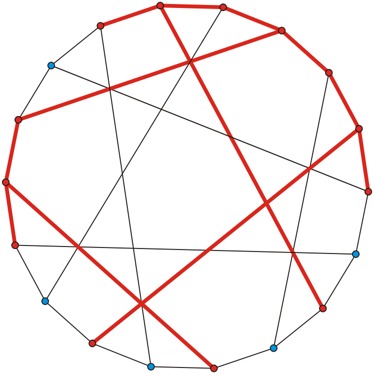


F16

Independent sets and their complements in cubic graphs



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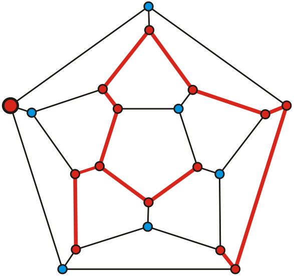

F20A

Payan, Sakarovitch, 1975

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Let X be a cyclically 4-edge-connected cubic graph of order n, and let S be a maximum cyclically stable subset of $V(X)$. Then $|S| = \lfloor (3n-2)/2 \rfloor$ and more precisely, the following hold.

- If $n \equiv 2 \,(mod\, 4)$ then $|S| = (3n-2)/4$, and $X[S]$ is a tree and $V(X) \setminus S$ is an independent set of vertices;
- If $n \equiv 0 \,(mod\, 4)$ then $|S| = (3n-4)/4$, either $X[S]$ is a tree and $V(X) \setminus S$ induces a graph with a single edge, or $X[S]$ has two components and $V(X) \setminus S$ is an independent set of vertices.

---

a cyclically stable subset = induces a forest

III. Non-existence of snarks amongst $(2, s, 3)$-Cayley graphs

A $(2, s, t)$-generated group is a group $G = \langle a, x \mid a^2 = x^s = 1, (ax)^t = 1, \dots \rangle$.

A $(2, s, t)$-Cayley graph is a cubic Cayley graph on G wrt $S = \{a, x, x^{-1}\}$.

Glover, KK, Malnič, Marušič, 2007-11

A $(2, s, 3)$-Cayley graph has

- a Hamilton cycle when $|G|$ is congruent to 2 modulo 4,
- a Hamilton cycle when $|G| \equiv 0 \pmod 4$ and either s is odd or $s \equiv 0 \pmod 4$, and
- a cycle of length $|G| - 2$, and also a Hamilton path, when $|G| \equiv 0 \pmod 4$ and $s \equiv 2 \pmod 4$.

Corollary

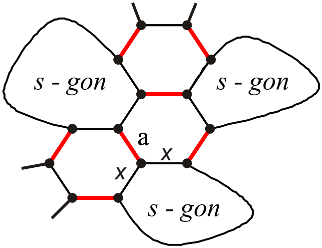
There are no snarks amongst $(2, s, 3)$-Cayley graphs.

Proof strategy

To a $(2, s, 3)$-Cayley graph X we associate a Cayley map $M(X)$ of genus

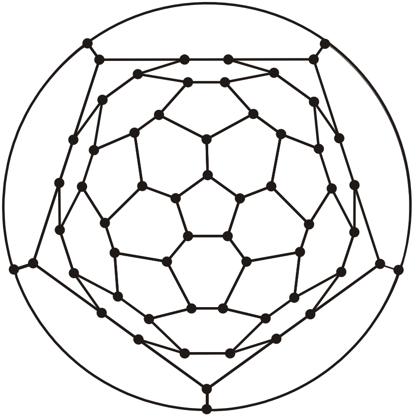
$$1 + (s - 6)|G|/12s$$

with faces $|G|/s$ disjoint s-gons and $|G|/3$ hexagons.



Soccer ball

(2, 5, 3)-Cayley graph of $A_5 = \langle a, x \mid a^2 = x^5 = (ax)^3 = 1 \rangle$.



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To X we associate a 'quotient graph', the so-called hexagon graph $Hex(X)$, whose vertices are hexagons in $M(X)$ with adjacencies arising from neighboring hexagons.

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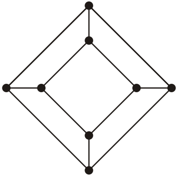
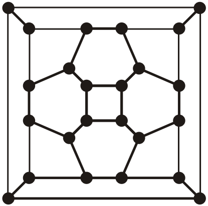
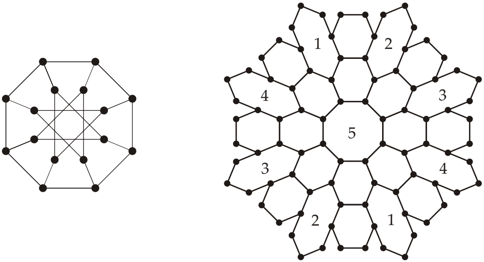




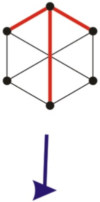
(2, 4, 3)-Cayley graph of $S_4 = \langle a, x \mid a^2 = x^4 = (ax)^3 = 1\rangle$.

Proof strategy

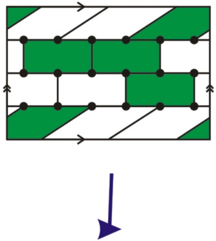


(2, 8, 3)-Cayley graph of $Q_8 \rtimes S_3$.

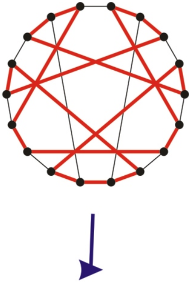
A $(2, 6, 3)$-generated group
$S_3 \times \mathbb{Z}_3 \cong \langle a, x \mid a^2 = x^6 = (ax)^3 = 1, \ldots \rangle$, where $a = ((12), 0)$ and $x = ((13), 1)$.



The corresponding hexagon graph.

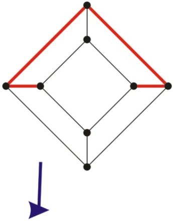


A Hamilton tree of hexagons.

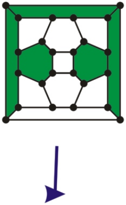


The corresponding Hamilton cycle in X.

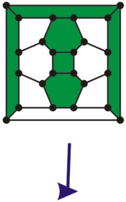
A $(2, 4, 3)$-generated group $S_4 \cong \langle a, x \mid a^2 = x^4 = (ax)^3 = 1 \rangle$, where $a = (12)$ and $x = (1234)$.



The same tree in the corresponding hexagon graph

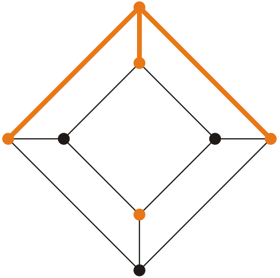


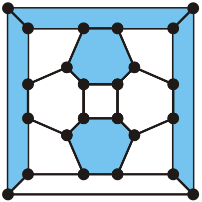
A tree of hexagons, whose boundary is a cycle missing only two vertices in the spherical Cayley map of X.



A modified tree of faces (including also a square).

A $(2,4,3)$-generated group $S_4 \cong \langle a, x \mid a^2 = x^4 = (ax)^3 = 1 \rangle$, where $a = (12)$ and $x = (1234)$.





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For cubic arc-transitive graphs (other then K_4) a result of Nedela and Škoviera (1995) implies cyclic 4-edge connectivity and so Payan-Sakarovitch theorem holds.

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Proposition

Let Y be a cubic arc-transitive graph. Then one of the following occurs.

- The girth $g(Y)$ of Y is at least 6; or
- Y is one of the following graphs: the theta graph θ_2, K_4, $K_{3,3}$, the cube Q_3, the Petersen graph $GP(5,2)$ or the dodecahedron graph $GP(10,2)$.

No snarks amongst $(2, s, 3)$-Cayley graphs

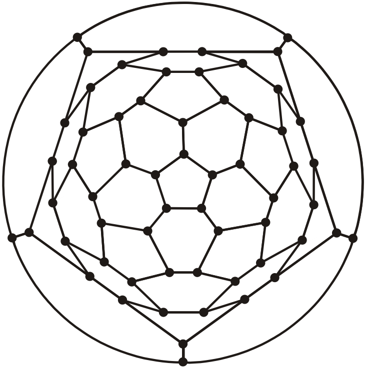
Corollary of Payan-Sakarovitch result for graphs

Let X be a cyclically 4-edge connected cubic graph of order $n \equiv 0 \pmod 4$. Then there exists a cyclically stable subset S of $V(X)$ such that $X[S]$ is a forest and $V(X) \setminus S$ is an independent set of vertices.

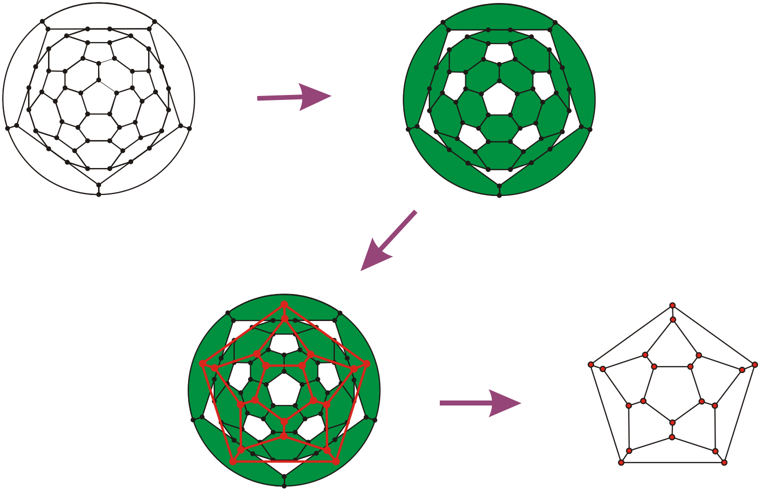
$\Rightarrow$ There are no snarks amongst $(2, s, 3)$-Cayley graphs. □

Example of a $(2, s, 3)$-Cayley graph

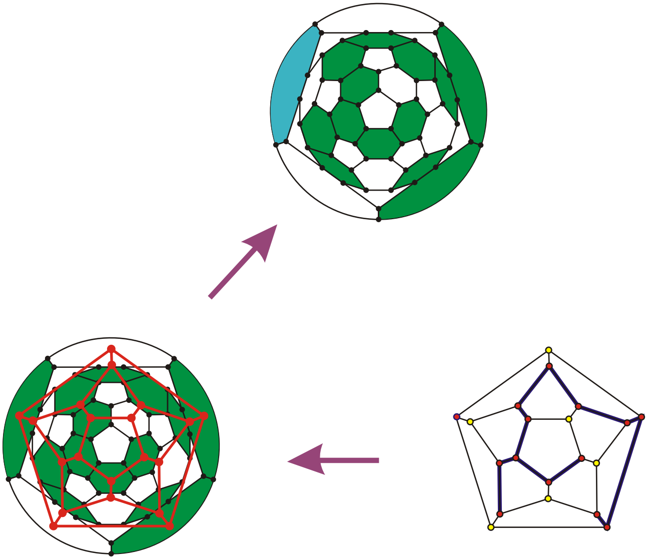
$X = Cay(A_5, \{a, x\})$ where $a = (12)(34)$ and $x = (12345)$ ($s = 5$, genus$= 0$).



From X to $Hex(X)$



From $Hex(X)$ to X



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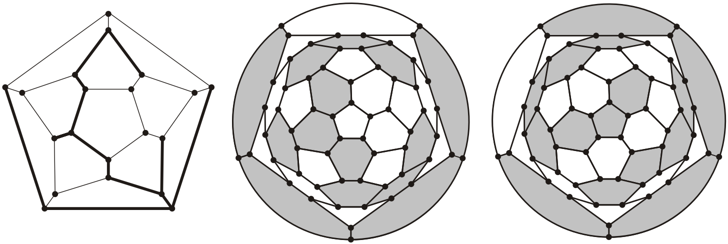

IV. Snarks and $(2, s, t)$-Cayley graphs

Method for $(2, s, t)$-Cayley graphs, $t > 3$

To a $(2, s, t)$-Cayley graph X we associate a Cayley map $M(X)$ with s-gonal and $2t$-gonal faces.

Further, to X we associate a 'quotient graph', the so-called $2t$-gonal graph X_{2t}, whose vertices are $2t$-gons in $M(X)$ with adjacencies arising from neighboring $2t$-faces. Note that X_{2t} is a t-valent arc-transitive graph admitting a 1-regular subgroup with a cyclic vertex-stabilizer $\mathbb{Z}_t$.

Sufficient conditions (in X_{2t}) for hamiltonicity / 3-edge colorability of X:

- If the vertex set V of X_{2t} decomposes into $(I, V - I)$ with I independent set and $V - I$ induces a tree then X contains a Hamiltonian cycle.
- If the vertex set V of X_{2t} decomposes into $(I, V - I)$ with I independent set and $V - I$ induces a bipartite graph then X is 3-edge colorable. (If X_{2t} is near-bipartite than X is not a snark.)

$(2, s, 4)$-Cayley snarks

Non-near-bipartite tetravalent arc-transitive graphs admitting a 1-regular subgroup with a cyclic vertex-stabilizer $\mathbb{Z}_4$: K_5, octahedron, $Cay(\mathbb{Z}_{13}, \{\pm 1, \pm 5\})$.

$(2, s, 4)$-Cayley snarks

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Are there other such graphs?

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Are there other such graphs?

Heuberger, Discrete Math, 2003

Let $X = C_n(a, b)$ be a tetravalent circulant of order n then

$$\chi(X) = \begin{cases} 2 & \text{if } a \text{ and } b \text{ are odd and } n \text{ is even} \\ 4 & \text{if } 3 \nmid n,\ n \neq 5, \text{ and } (b \equiv \pm 2a \pmod{n}) \text{ or } a \equiv \pm 2b \pmod{n}) \\ 4 & \text{if } n = 13 \text{ and } (b \equiv \pm 5a \pmod{1}3 \text{ or } a \equiv \pm 5b \pmod{1}3) \\ 5 & \text{if } n = 5 \\ 3 & \text{otherwise} \end{cases}$$

Problem

Classify tetravalent arc-transitive graphs with chromatic number 4 admitting a 1-regular subgroup with a cyclic vertex-stabilizer $\mathbb{Z}_4$.

PhD and Postdoc Summer School in Discrete Mathematics

June 24 to June 30, 2012, Rogla, Slovenia

June 27, 2012:

SYGN 2012 Symmetries of Graphs and Networks (Banff 3)

Computers in Scientific Discovery 6 (CSD6)

August 21 to August 25, 2012, Portorož, Slovenia
http://csd6.imfm.si, csd6@upr.si

The list of keynote speakers includes:
Nobelist Harold Kroto (not confirmed), Gunnar Brinkmann, Arnout Ceulemans, Ernesto Estrada, Patrick Fowler, Ante Graovac, Bojan Mohar, Dragan Stevanović, Ian Wanless, Jure Zupan.

Announcement 3

ARS MATHEMATICA CONTEMPORANEA

Proceedings of

Bled'11 - 7th Slovenian International Conference on Graph Theory

The deadline for submission is November 30, 2011.

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Thank you!