GEOMETRIC (n_k) CONFIGURATIONS

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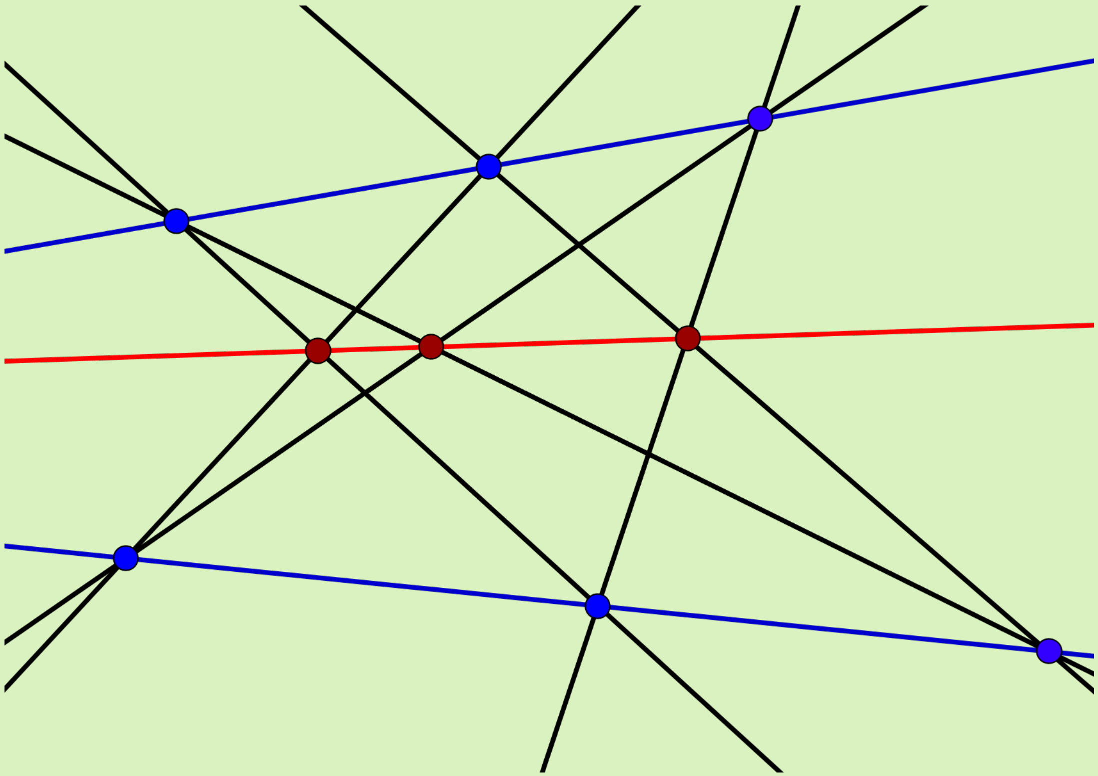
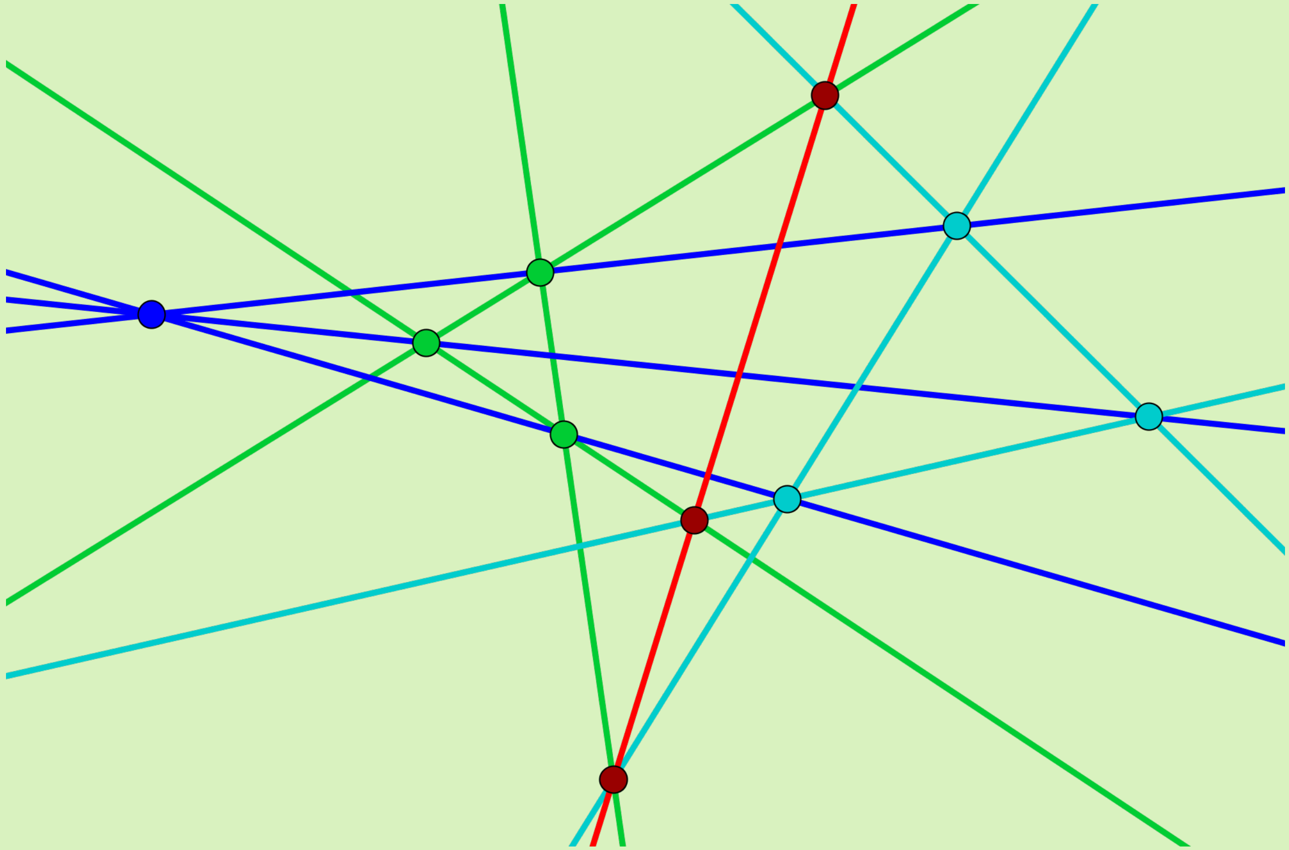


Pappus, 4th century A.D.

(9_3)



Desargues, 1648

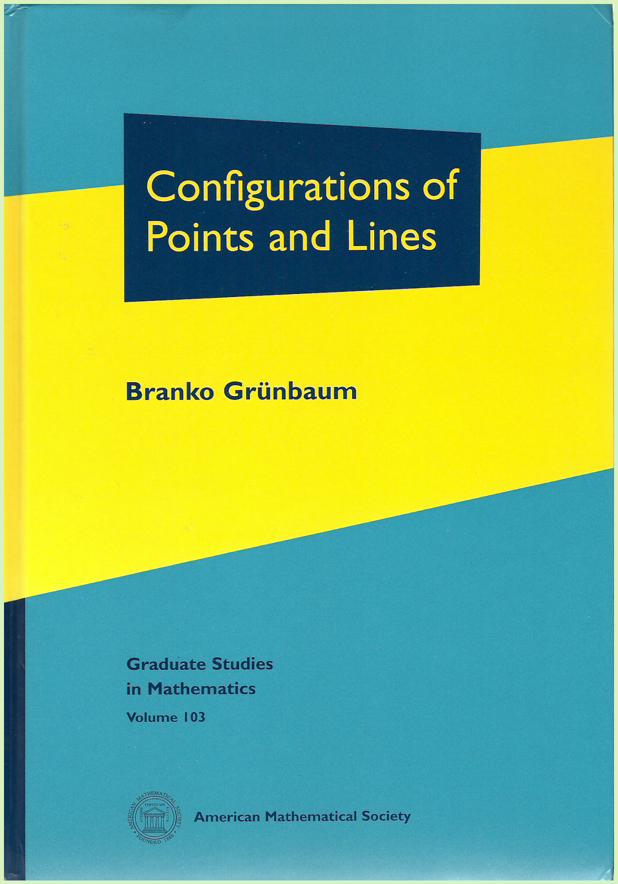
(10_3)

- A *geometric point-line configuration* (p_q, n_k) is a family of p points and n lines such that each point is incident with precisely q lines and each line is incident with precisely k points.

- If $p = n$ (hence $q = k$), then the configuration is called *balanced*. Notation: (n_k).

- The ambient space is $\mathbb{E}^d$ (or, the projective space $\mathbb{P}^d$). (We do not restrict ourselves to $\mathbb{E}^2$ or $\mathbb{P}^2$.)

From the 1990-ties there is a “renaissance” of configurations.



Branko Grünbaum (2009)

Instead of lines, one can also use: planes, hyperplanes, circles, ellipses

↓

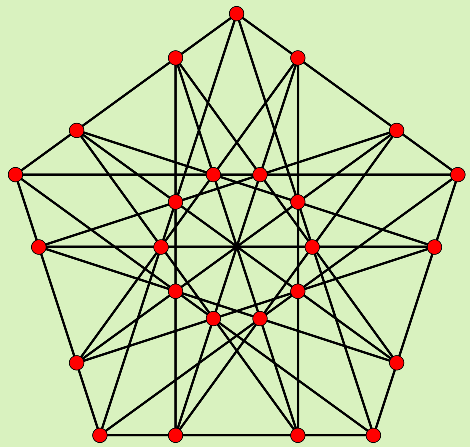
- point-plane;
- point-hyperplane;
- point-circle;
- point-ellipse configurations.

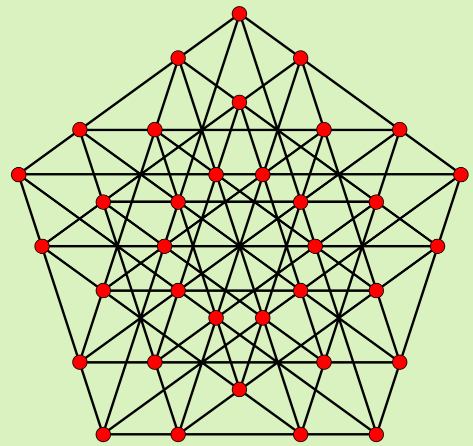
SPATIAL POINT-LINE CONFIGURATIONS

Construction principle:

- *highly symmetric convex polytopes can serve as a scaffolding for building large spatial configurations*;
- *the configuration inherits the symmetry of the polytope*.

Configurations with the symmetry of a Platonic solid





Jürgen Bokowski's
(25_4)
↓
(540_4)

Branko Grünbaum's
(35_4)
↓
(780_4)

(Supporting polytope: pentagonal dodecahedron)

Configurations with the symmetry of a Platonic solid

Three infinite series (for $t = 1, 2, \ldots$) :

- tetrahedron $\longrightarrow$ $\big((18(t+1)_3\big)$
- cube $\longrightarrow$ $\big((36(t+1)_3\big)$
- icosahedron $\longrightarrow$ $\big((90(t+1)_3\big)$

An example with chiral symmetry: (180_3).

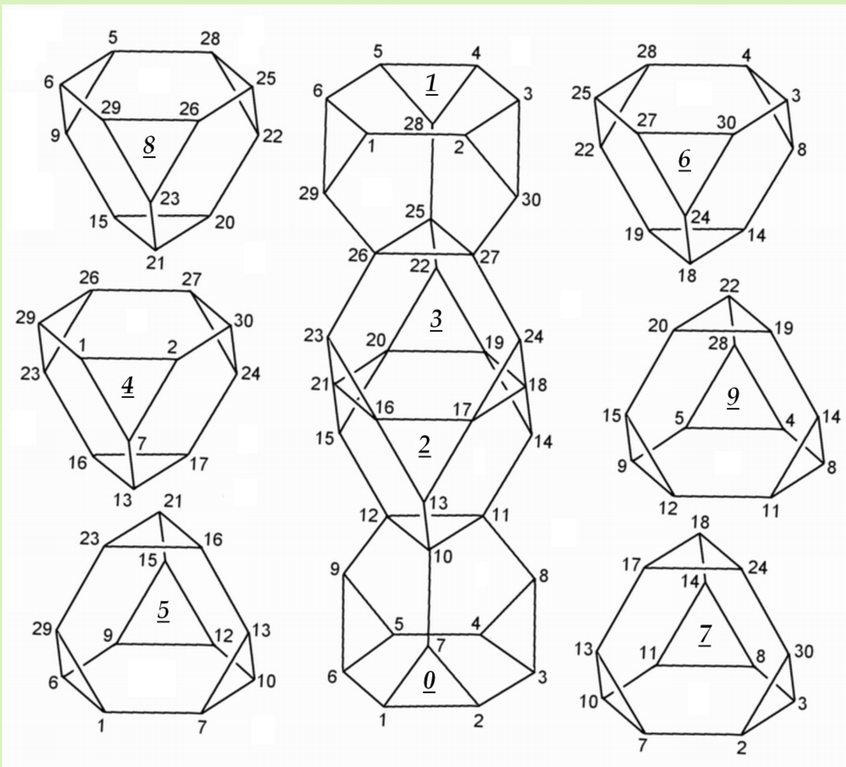
Configurations with the symmetry of a regular 4-polytope

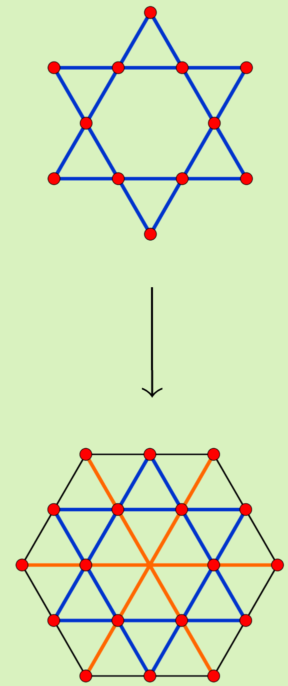
- regular 4-simplex $(|G| = 120) \longrightarrow (240_3)$
- 4-cube $(|G| = 384) \longrightarrow (768_3)$
- regular 120-cell $(|G| = 14400) \longrightarrow (28800_3)$

(n_3), where $n = 2 \times$ (order of the symmetry group);

7 orbits of points, 5 orbits of lines.

Configurations with the symmetry of other 4-polytopes

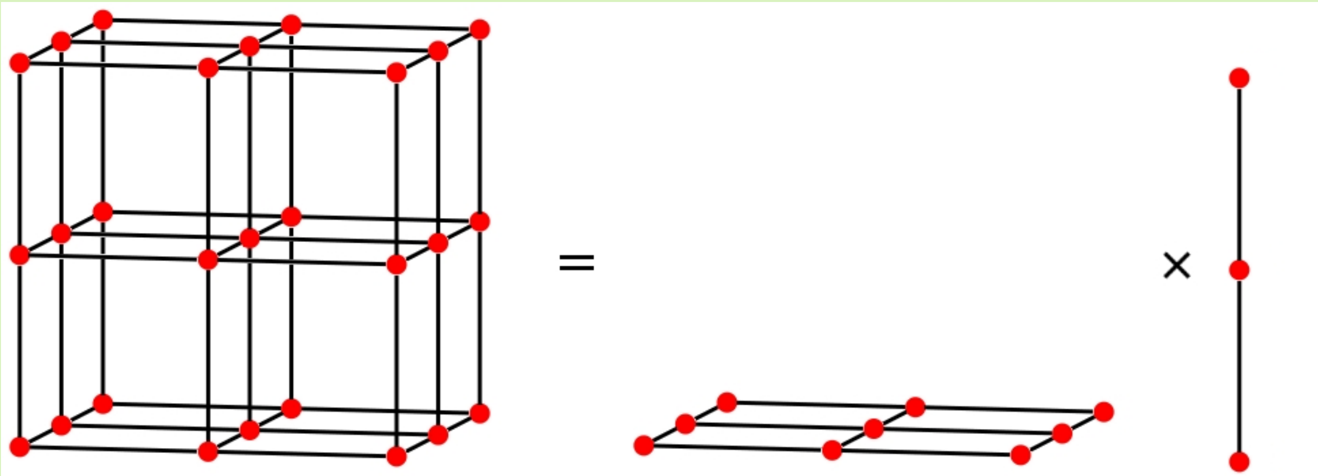




Uniform 10-cell + $(12_2, 6_4) \longrightarrow (420_4)$ $(|G| = 240)$

Uniform 48-cell + $(16_2, 8_4) \longrightarrow (4032_4)$ $(|G| = 2304)$

Cartesian product of point-line configurations



27_3
Gray configuration

$(9_2, 6_3)$

$(3_1, 1_3)$

Cartesian product of point-line configurations

Definition.
Let $\mathcal{C}_1$ be a (p_q, m_k) configuration in an Euclidean space $\mathbb{E}_1$ and $\mathcal{C}_2$ be an (r_s, n_k) configuration in an Euclidean space $\mathbb{E}_2$. Observe that these two configurations have the same number k of points on each line. The Cartesian product of $\mathcal{C}_1$ and $\mathcal{C}_2$ is the

$$\big((pr)_{(q+s)}, (pn+rm)_k\big)$$

configuration $\mathcal{C}_1 \times \mathcal{C}_2$ in $\mathbb{E}_1 \times \mathbb{E}_2$ whose point set is the Cartesian product of the point sets of $\mathcal{C}_1$ and $\mathcal{C}_2$ and where there is a line incident to two points (x_1, x_2) and (y_1, y_2) if and only if either $x_1 = y_1$ and there is a line incident to x_2 and y_2 in $\mathcal{C}_2$, or $x_2 = y_2$ and there is a line incident to x_1 and y_1 in $\mathcal{C}_1$.

Powers of configurations

- complete 5-lateral: $(10_2, 5_4)^2 = (100_4) \subset \mathbb{P}^4$
- complete 7-lateral: $(21_2, 7_6)^3 = (9261_6) \subset \mathbb{P}^6$
- complete 9-lateral: $(36_2, 9_8)^4 = (1\ 679\ 616_8) \subset \mathbb{P}^8$
- complete 11-lateral: $(55_2, 11_{10})^5 = (503\ 284\ 375_{10}) \subset \mathbb{P}^{10}$

$\vdots$

- complete $(2k+1)$-lateral:

$$\left(\binom{2k+1}{2}_2, (2k+1)_{2k} \right)^k = \left(\left(\binom{2k+1}{2}^k \right)_{2k} \right) \subset \mathbb{P}^{2k}$$

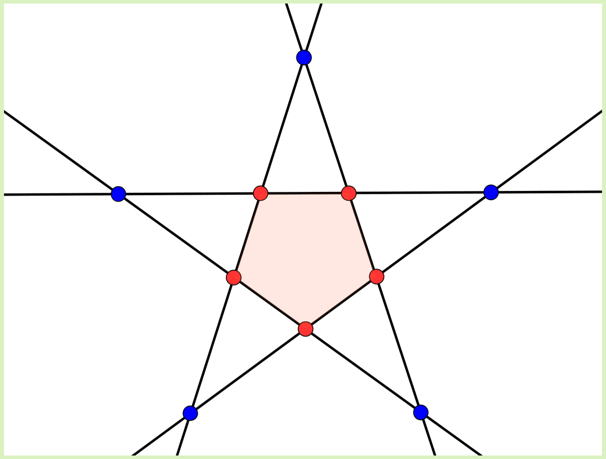
Powers of configurations

Scaffolding polytope: *rhombicosidodecahedron*.

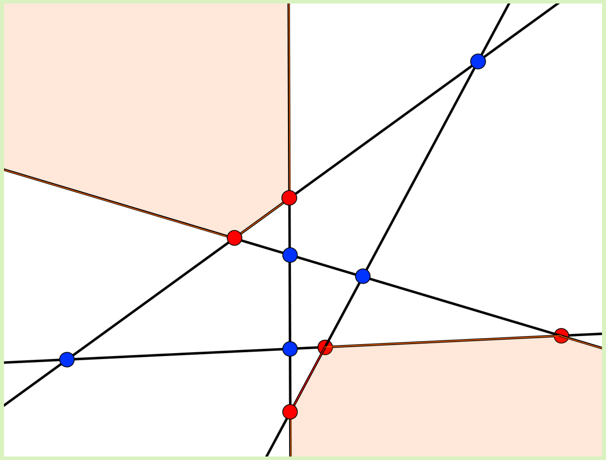
- $$(120_2, 60_4)^2 = \left(14400_4\right) \subset \mathbb{E}^6$$
- $$(180_2, 60_6)^3 = \left(5832200_6\right) \subset \mathbb{E}^9$$
- $$(240_2, 60_8)^4 = \left(3\ 317\ 760\ 000_8\right) \subset \mathbb{E}^{12}$$
- $$(300_2, 60_{10})^5 = \left((2.43 \cdot 10^{10})_{10}\right) \subset \mathbb{E}^{15}$$
- $$(360_2, 60_{12})^6 = \left((2^{18} \cdot 3^{12} \cdot 5^6)_{12}\right) \subset \mathbb{E}^{18}$$

An incidence statement on complete pentalaterals

Lemma. *The set of vertices of a complete pentalateral* $P(l_1, \ldots, l_5)$ *can be uniquely partitioned to "external" and "internal" vertices.*



"Symmetric" position



"General" position

Let be given 25 lines, a_{ij} $(i,j=1,\ldots,5)$, in the projective space $\mathbb{P}^3$ such that they form five complete pentalaterals:

$$A_1 = P(a_{11},\ldots,a_{15}),\ \ldots,\ A_5 = P(a_{51},\ldots,a_{55}).$$

Assume that the following conditions hold:

1. the external vertices of the pentalaterals A_i form the external vertices of complete pentalaterals $B_j = P(b_{1j},\ldots,b_{5j})$, as follows:

$$a_{ij} \cap a_{i,j+2} = b_{ij} \cap b_{i+2,j};$$

2. the internal vertices of the pentalaterals A_i form the external vertices of complete pentalaterals $C_j = P(c_{1j},\ldots,c_{5j})$, as follows:

$$a_{ij} \cap a_{i,j+1} = c_{ij} \cap c_{i+2,j}$$

(indexing is meant modulo 5).

Then there is a quintuple of complete pentalaterals D_i such that their vertices coincide with the internal vertices of the pentalaterals B_j and C_j, as follows:

$$b_{ij} \cap b_{i+1,j} = d_{ij} \cap d_{i,j+2} \quad \text{and} \quad c_{ij} \cap c_{i+1,j} = d_{ij} \cap d_{i,j+1}.$$

An incidence statement on complete pentalaterals

Some facts supporting the statement:

- *There exists a balanced configuration* (100_4) *in* $\mathbb{P}^3$ *such that its points are just the points of intersection satisfying the four conditions of the statement.*

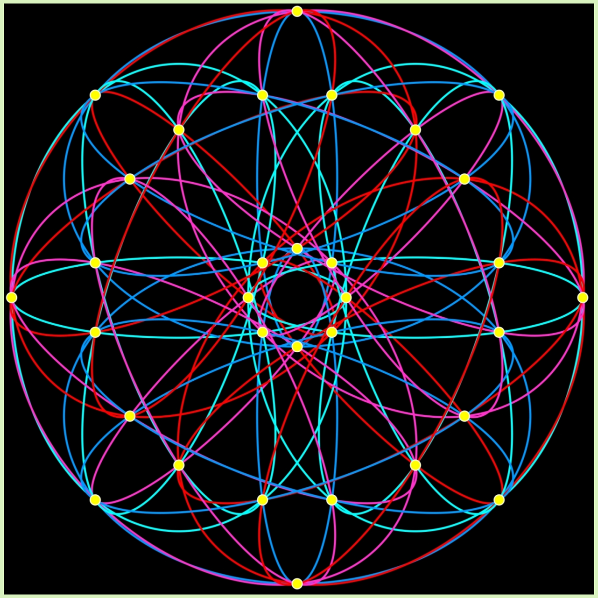
- (Special case in $\mathbb{E}^3$ when the statement holds.)

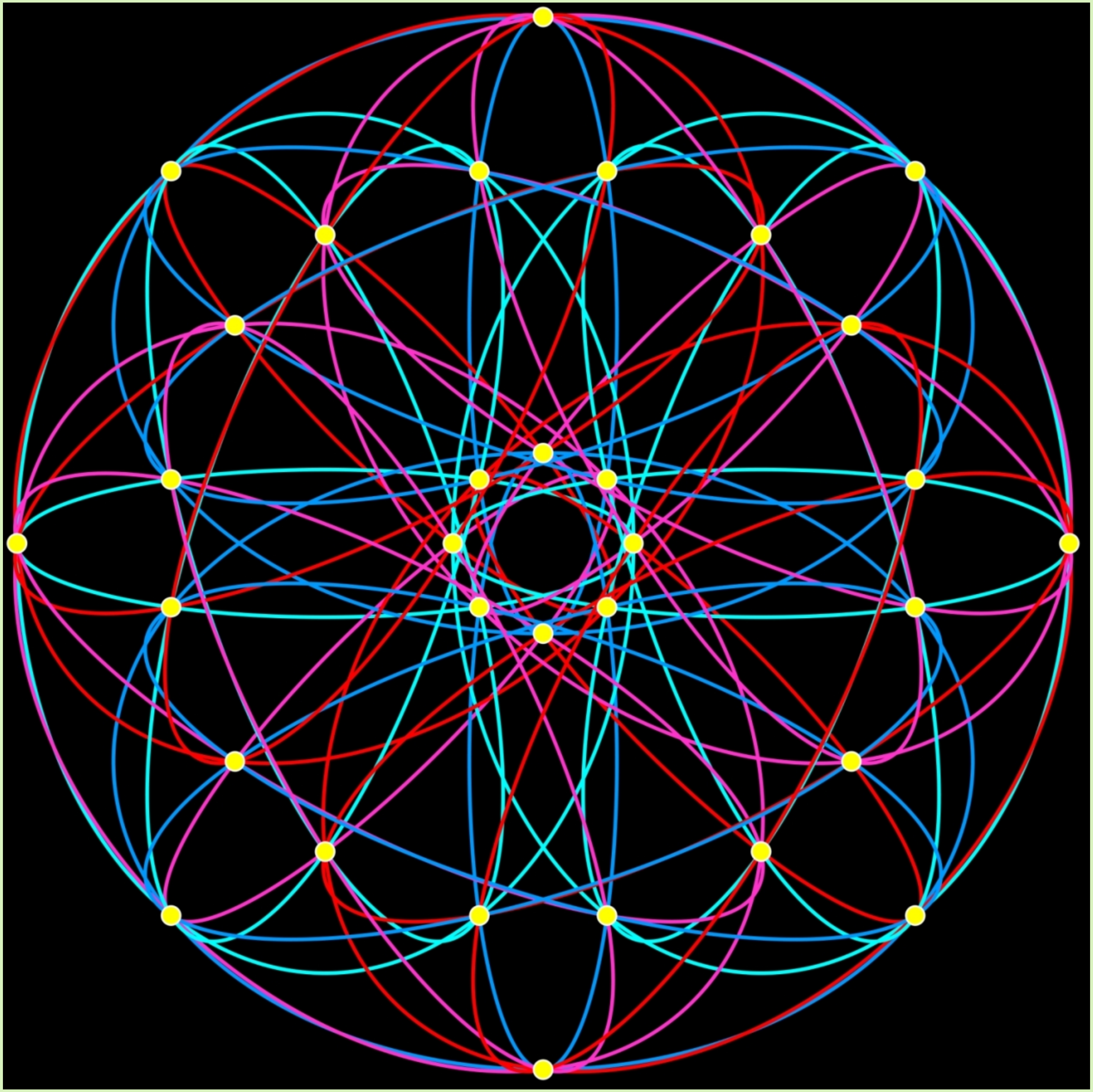
  The pentagons determined by the A_is and D_is are all regular (in Euclidean sense), and they have a common axis of rotation (of order five). In this case the conditions of the conjecture can easily be satisfied by suitably scaling the A_is and D_is and by suitably chosen shapes and sizes of the B_js and D_js.

- (Special case in $\mathbb{E}^3$ when the statement is supported by simulation.)

  All the pentalaterals A_i are homothetic copies of a pentalateral A_0. Furthermore, the external vertices of A_0 (hence those of each A_i) are inscribed in a circle. (Modelled in *Mathematica* by Karsai and Szilassi, 2008).

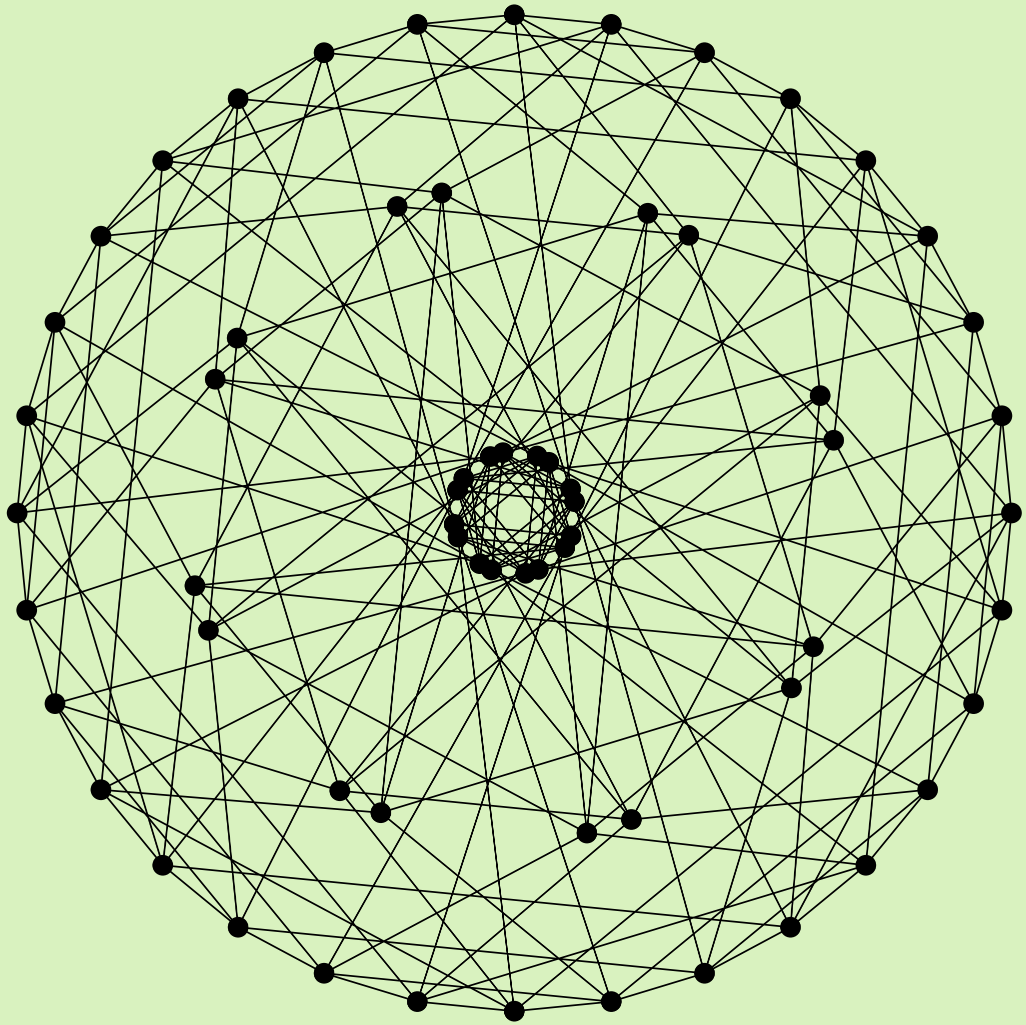
POINT-ELLIPSE CONFIGURATIONS





(32_6)

GG (2009)



The Levi graph of the (32_6) point-ellipse configuration
(by Tomaž Pisanski)

An analogus version: (96_6), derived from the regular 24-cell.

From a different point of view, the (32_6) configuration can be considered as the starting member of an infinite series of point-ellipse configurations $\mathcal{C}_n$, whose type is

$$\big((2n^2)_6\big), \quad (n = 4, 5, \ldots).$$

The sketch of the construction:

- take the Cartesian product of two equal regular n-gons with even $n \geq 4$; this is a 4-polytope with $2n$ prismatic facets;
- inscribe into these prisms affinely regular hexagons;
- circumscribe ellipses around these hexagons.

The set of the points of $\mathcal{C}_n$ is

$$\left\{ v_j^{i,i+1},\, v^i_{j,j+1} \,\middle|\, i,j \in [n] \right\},$$

and the inscribed hexagons are of the following forms:

$$\left\{ v^i_{j-1,j}, \underline{v_j^{i,i+1}}, v^{i+1}_{j,j+1}, v^{i+\frac{1}{2}n}_{j-1,j}, \underline{v_j^{i+\frac{1}{2}n,i+\frac{1}{2}n+1}}, v^{i+\frac{1}{2}n+1}_{j,j+1} \right\},$$

$$\left\{ v^i_{j,j+1}, \underline{v_j^{i,i+1}}, v^{i+1}_{j-1,j}, v^{i+\frac{1}{2}n}_{j,j+1}, \underline{v_j^{i+\frac{1}{2}n,i+\frac{1}{2}n+1}}, v^{i+\frac{1}{2}n+1}_{j-1,j} \right\},$$

$$\left\{ v_j^{i-1,i}, \underline{v^i_{j,j+1}}, v_{j+1}^{i,i+1}, v_{j+\frac{1}{2}n}^{i-1,i}, \underline{v^i_{j+\frac{1}{2}n,j+\frac{1}{2}n+1}}, v_{j+\frac{1}{2}n+1}^{i,i+1} \right\},$$

$$\left\{ v_j^{i,i+1}, \underline{v^i_{j,j+1}}, v_{j+1}^{i-1,i}, v_{j+\frac{1}{2}n}^{i,i+1}, \underline{v^i_{j+\frac{1}{2}n,j+\frac{1}{2}n+1}}, v_{j+\frac{1}{2}n+1}^{i-1,i} \right\},$$

for all $i, j \in [n]$.

POINT-PLANE CONFIGURATIONS

"V-construction": (GG, 2009)

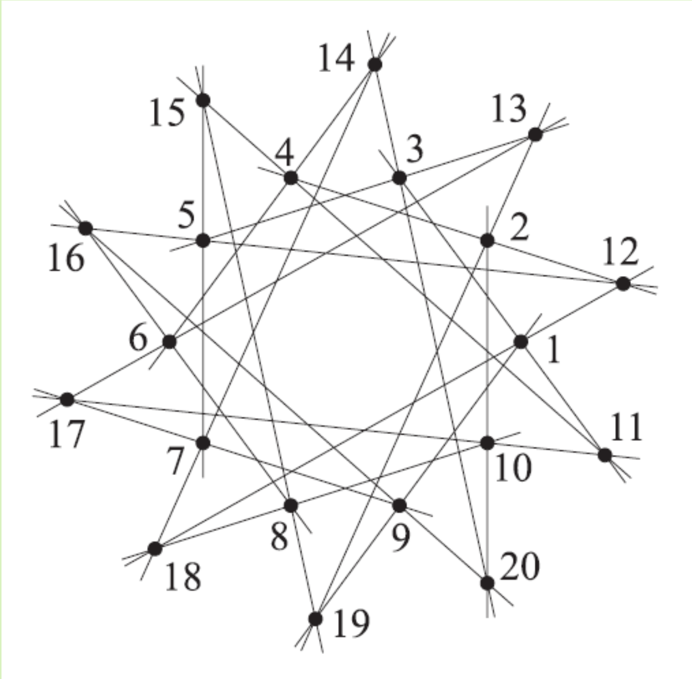
- start from a highly symmetric polytope P (e.g. Platonic solid, Archimedean solid, etc.);
- the points of the configuration $V(P)$ are the vertices of P;
- for each vertex v of P, take the plane spanned by the vertices which are the first neighbours of v.

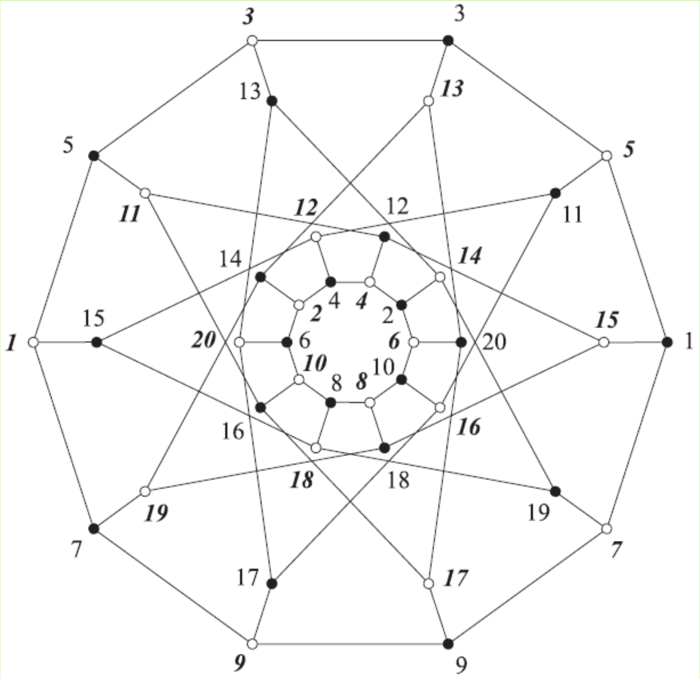
A direct consequence of the construction:

The Levi graph of the configuration $V(P)$ *is isomorphic to the Kronecker cover of the 1-skeleton of the polytope* P:

$$L(V(P)) \cong KC(G(P))$$

(A graph $\widetilde{G}$ is said to be the Kronecker cover of the graph G if there exists a two-to-one surjective homomorphism $f : \widetilde{G} \to G$ such that for every vertex v of $\widetilde{G}$ the set of edges incident with v is mapped bijectively onto the set of edges incident with $f(v)$.)





The flag-transitive triangle-free point-line configuration (20_3),
and its Levi graph.
(This Levi graph is the Kronecker cover of the dodecahedron graph.)
[Boben, Grünbaum, Pisanski and Žitnik, 2006]

V-construction from Platonic solids

V_1(tetrahedron)	(4_3)
V_1(cube)	$2 \times (4_3)$
V_1(icosahedron)	(12_5)
V_1(dodecahedron)	(20_3)
V_2(dodecahedron)	(20_6)

V-construction from Archimedean solids

V(truncated tetrahedron)	(12_3)
V(cuboctahedron)	(12_4)
V(truncated octahedron)	(24_3)
V(truncated cube)	(24_3)
V(rhombicuboctahedron)	(24_4)
V(great rhombicuboctahedron)	(48_3)
V(truncated icosahedron)	(12_3)
V(truncated dodecahedron)	(20_3)
V(icosidodecahedron)	(30_4)
V(rhombicosidodecahedron)	(60_4)
V(great rhombicosidodecahedron)	(120_3)
V(snub cube)	(24_5)
V(snub dodecahedron)	(60_5)

V-construction from regular d-polytopes

$(d \geq 4)$

V_1(24-cell)	(24_8)
V_1(600-cell)	(120_{12})
V_2(600-cell)	(120_{20})
V_1(120-cell)	(600_4)
V_2(120-cell)	(600_{12})
V_1(d-simplex)	$((d+1)_d)$
V_1(d-cube)	$2 \times \left(\left(2^{d-1}\right)_d\right)$

A sufficient condition for applicability of the V-construction:

Assume that P is a polytope such that

1. P is vertex-transitive;
2. the stabilizer of each vertex v of P is transitive on the vertices adjacent to v.

Then the configuration $V(P)$ exists.

Corollary. $V(P)$ is self-polar.

Definition. A configuration $\mathcal{C}$ is self-polar if there exists a fixed-point-free and involutive automorphism π of $L(\mathcal{C})$ such that the bipartition of $L(\mathcal{C})$ is interchanged by π, and for any vertex v of $L(\mathcal{C})$ the vertices v and $\pi(v)$ are non-adjacent.

A hypersimplex $\Delta(d, k)$ is defined as a convex polytope whose vertices are given by vectors of length $d+1$ which consist of k ones and $d-k+1$ zeroes.
(Gelfand et al., 1975; Ziegler, 1995).

It can also be defined as the convex hull of the centroids of $(k-1)$-dimensional faces of a regular d-simplex.

For each $d \geq 4$ and $1 < k \leq d/2$, there exists the point-hyperplane configuration $V(\Delta(d, k))$ of type

$$\left(\binom{d+1}{k}_{k(d-k+1)} \right).$$

Let T and T' be two concentric regular simplices of equal size such that one is the mirror image of the other with respect to their common centre. Then the intersection $T \cap T'$ is called a uniform duplex. A polytope combinatorially equivalent to a uniform duplex called a duplex. (GG, 2009)

(A duplex of odd dimension is the hypersimplex $\Delta(d, (d+1)/2)$.)

Let D be a duplex of even dimension. For each $d \geq 4$, there exists the point-hyperplane configuration $V(D)$ of type

$$\left(\left((d+1)\binom{d+1}{k}\right)_d\right).$$

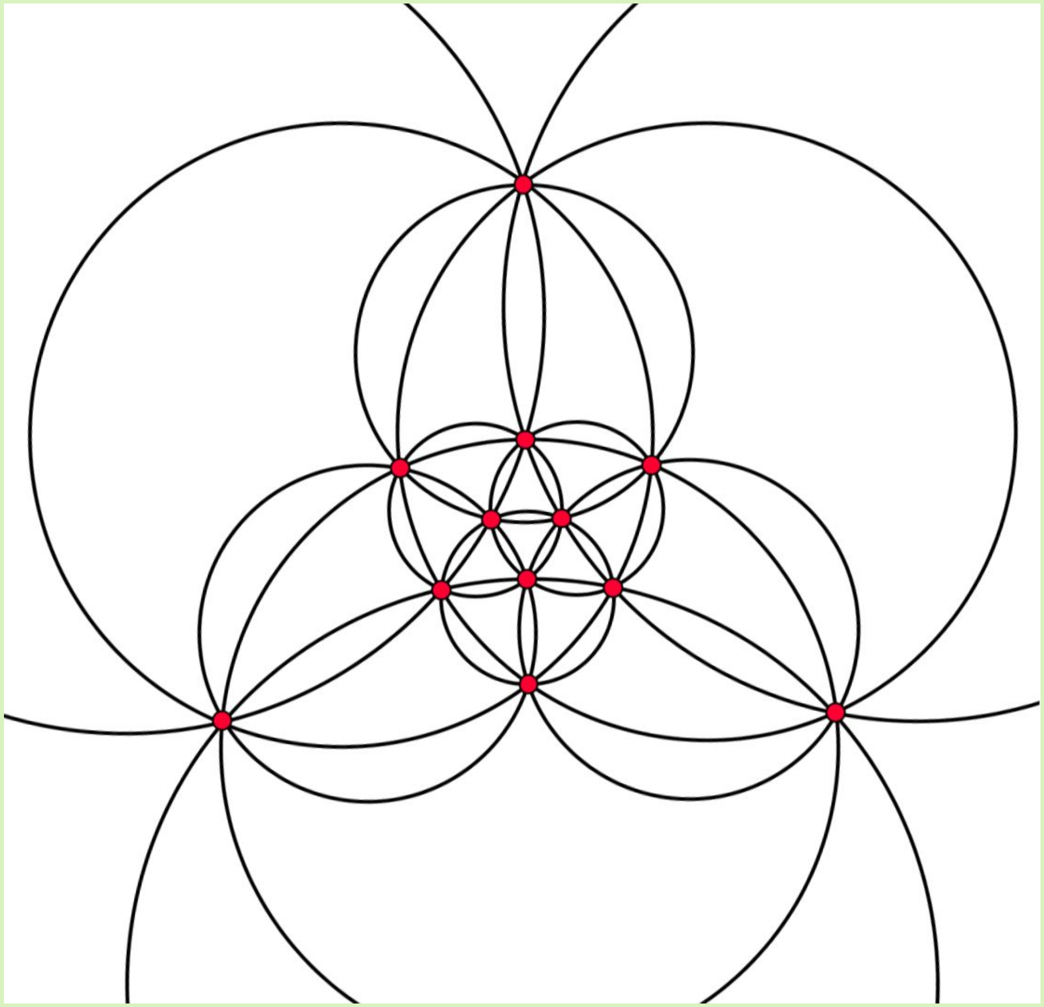
POINT-CIRCLE CONFIGURATIONS

The classical example: Clifford's chain

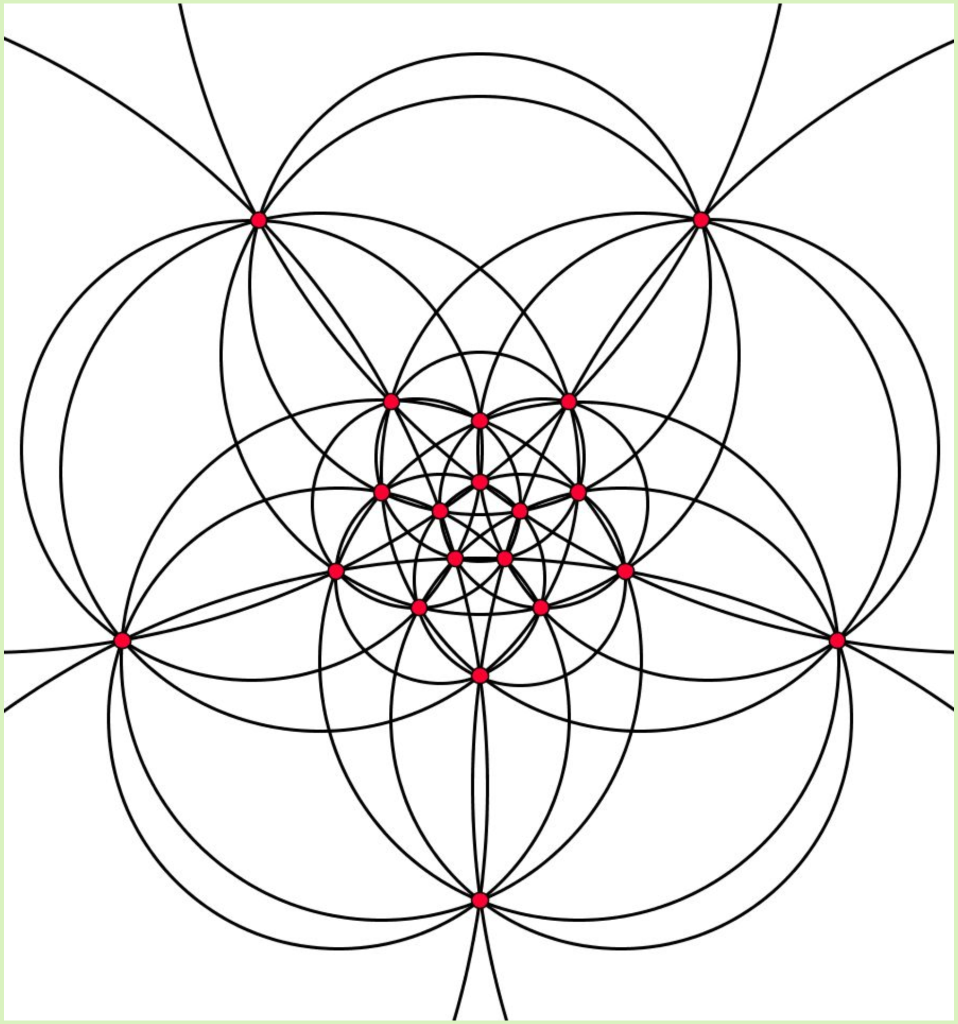
$$\left(\left(2^n\right)_{n+1}\right), \quad (n = 3, 4, \dots)$$

(Clifford, 1882; Coxeter, 1950, 1961)

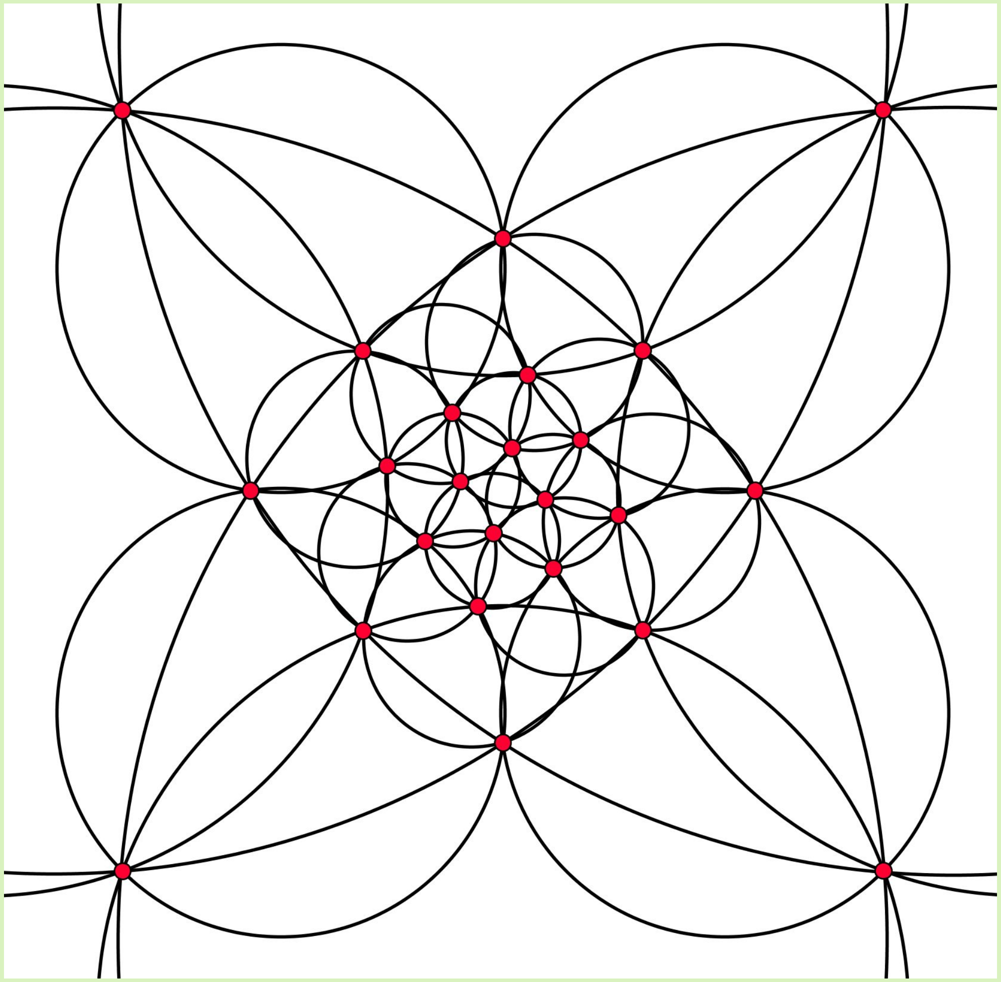
A new (limitedly movable) example: (12_4).



(12_5)



(20_6)

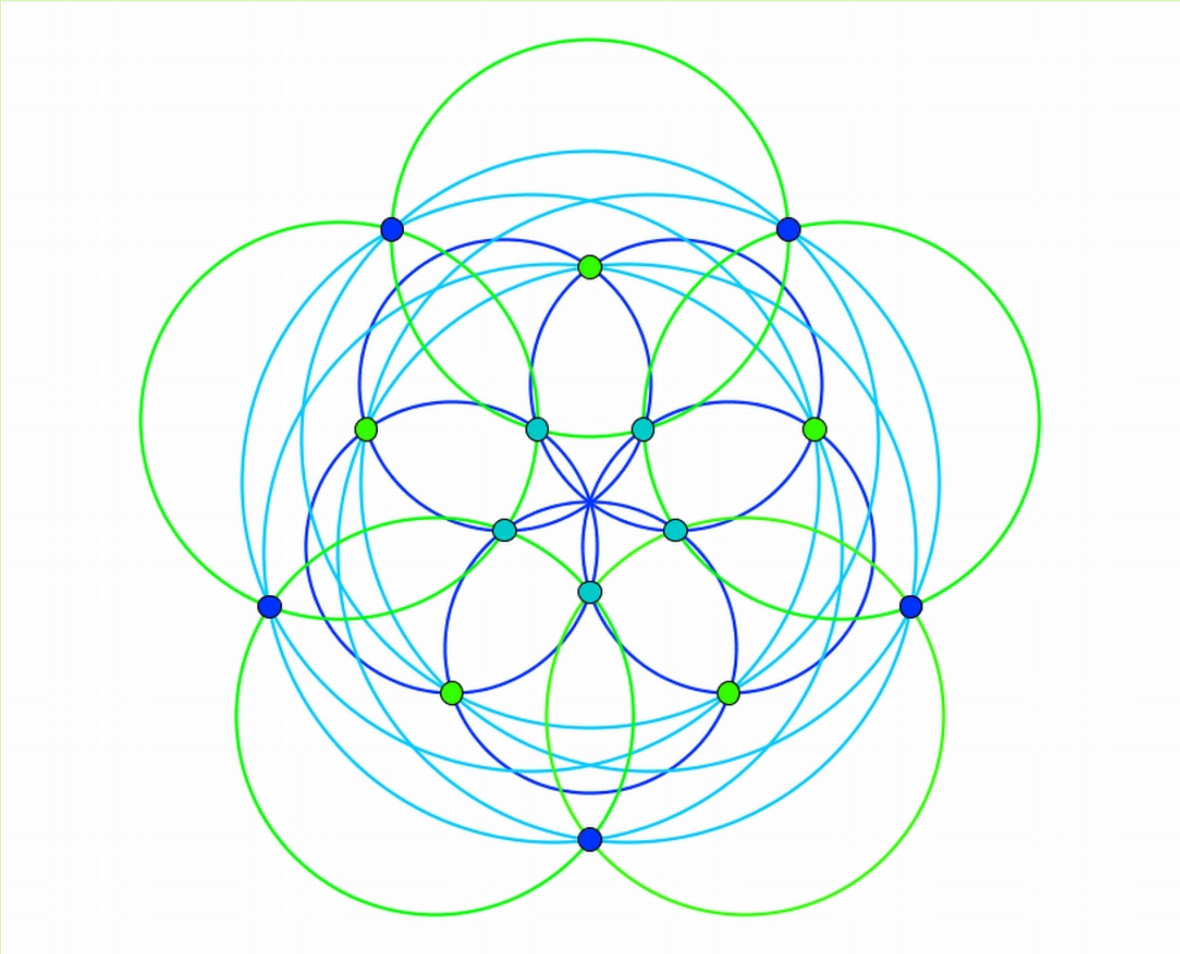


(24_5)

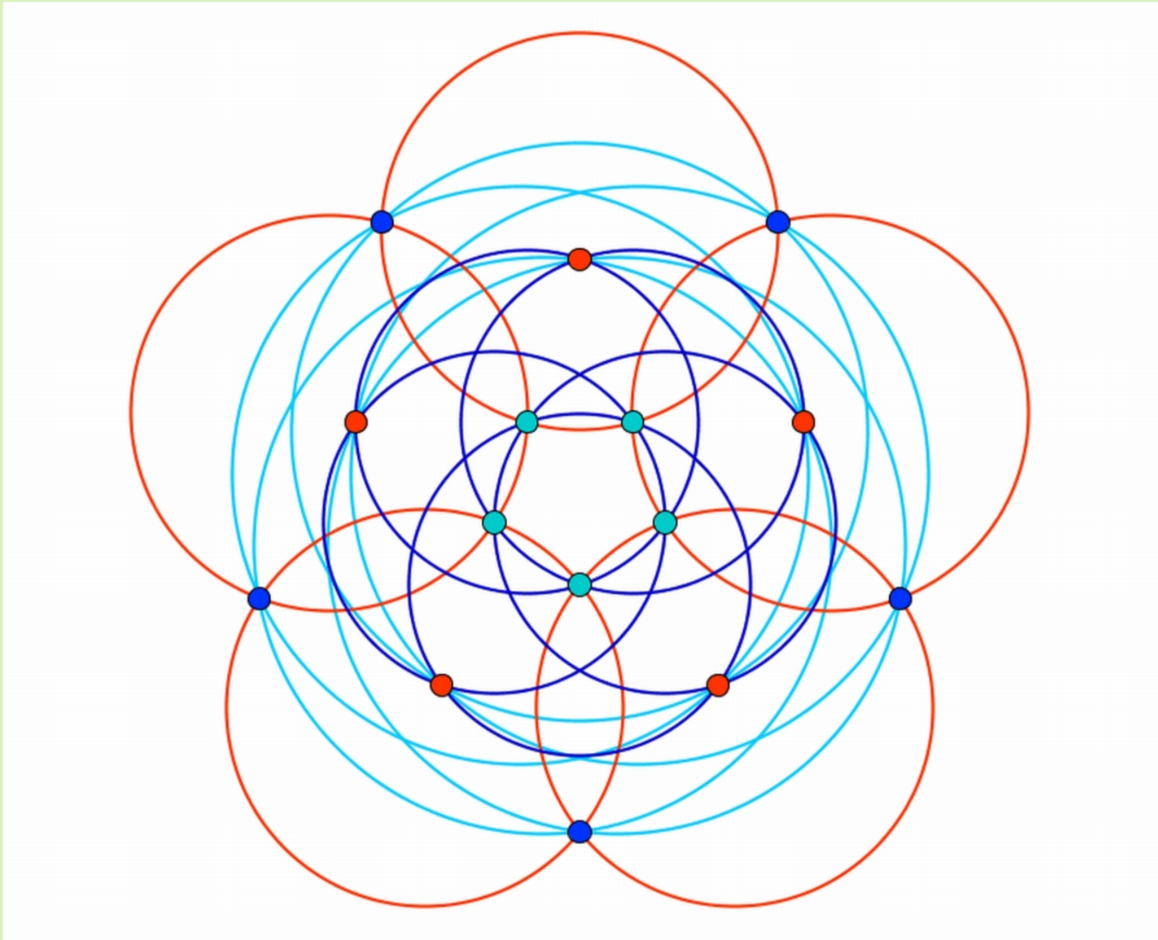
POINT-CIRCLE CONFIGURATIONS

For each integer $n \geq 3$, there is an infinite series of type

$$\big((3n)_4\big).$$



(15_4)



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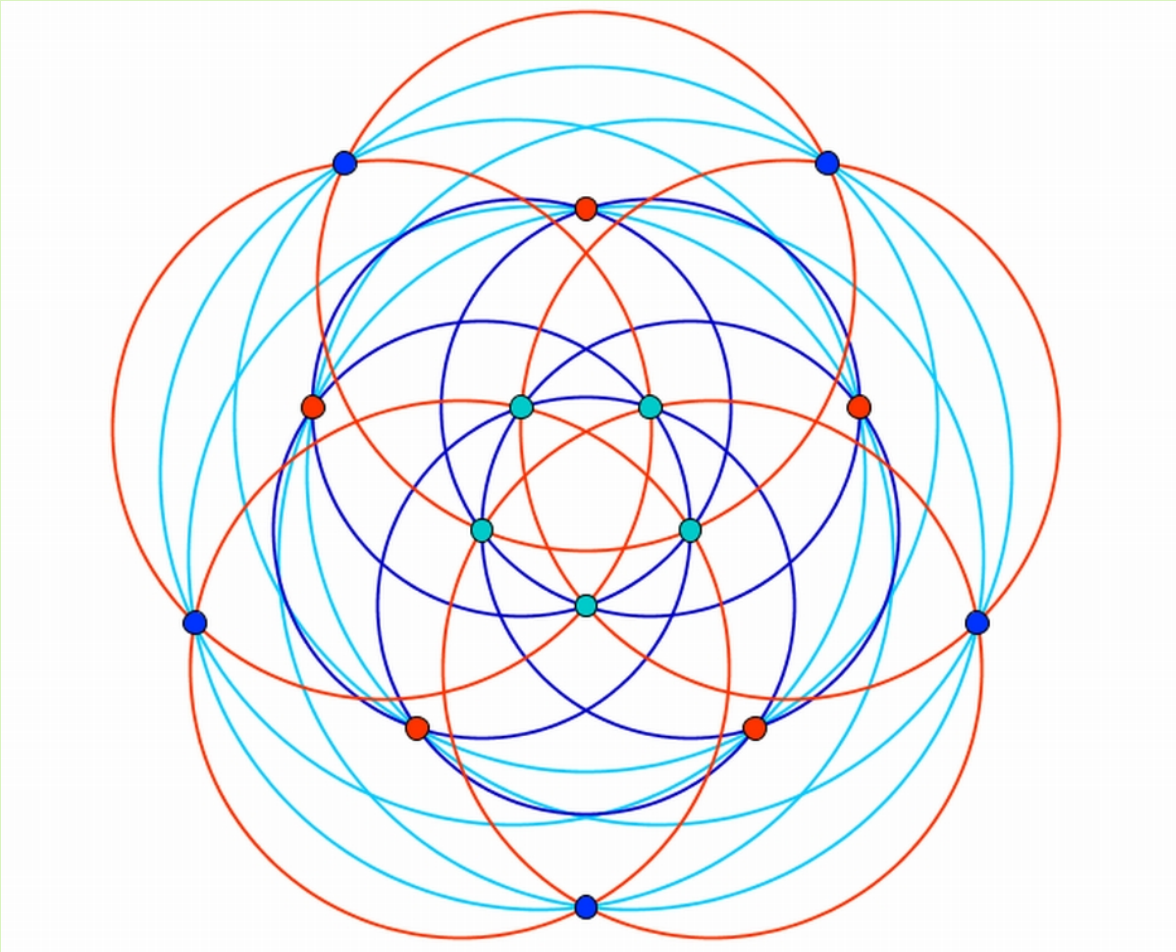

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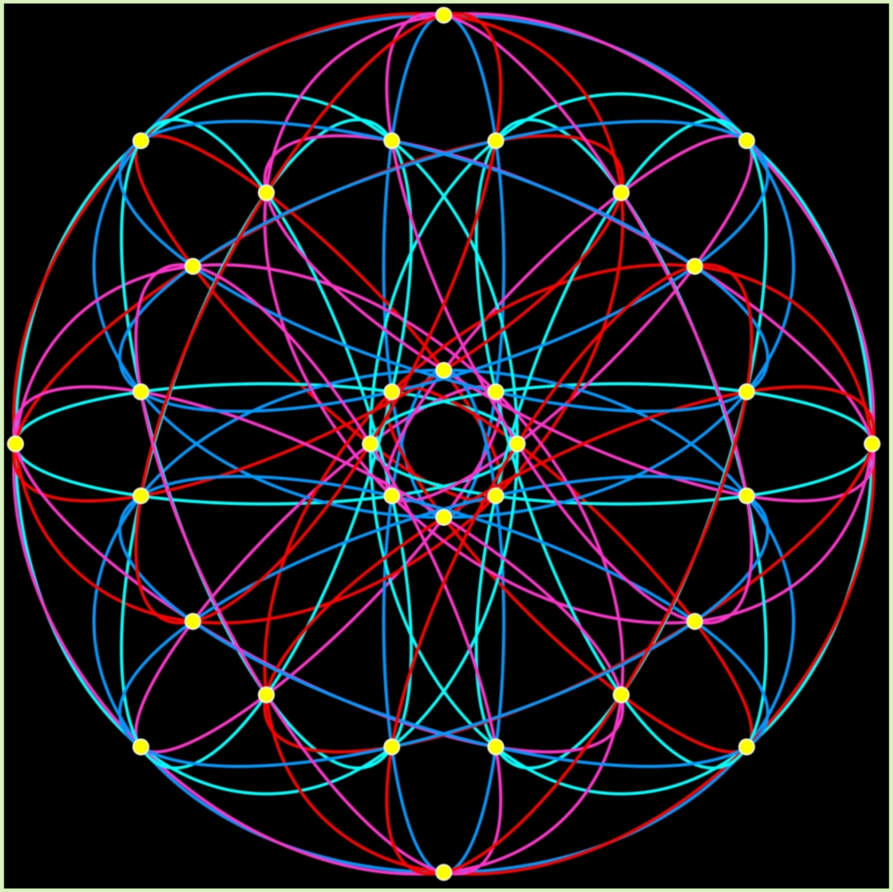

Thank you for your attention.