Graph Isomorphism problem, Weisfeiler-Leman algortihm and coherent configurations

M. Muzychuk

Netanya Academic College,
Israel

University of Primorska, Koper
Slovenia, February 2015

Content

- Graph isomorphism problem and Weisfeiler-Leman algorithm.
- Coherent configurations and coherent (cellular) algebras.
- Association schemes.

References. Coherent configurtion and algebras

[Hig70] D. G. Higman, Coherent configurations I, Rend. Sem. Mat. Univ. Padova 44(1970), 1-25

[Hig87] D. G. Higman,Coherent algebras, Linear Alg. and Its Applications 93(1987),209-239

[Wei76] B. Weisfeiler, On Construction and Identification of Graphs, LNM 558 (1976)

KRRT99] M. Klin, C. Rücker, G. Rücker and G. Tinhofer, Algebraic Combinatorics in Mahematical Chemistry. Methods and Algorithms. I. Permutation Groups and Coherent (Cellular) Algebras. Match (40), 1999, `shttp:\match.pmf.kg.ac.rs\electronic_versions\Match40\match40_7-138.pdf`

[EP09] S. Evdokimov and I. Ponomarenko, Permutation group approach to association schemes, Europ. J. of Combin. 30(2009), 1456-1476.

References. Association Schemes

[BI84] E. Bannai, T. Ito, Algebraic Combinatorics I: Association Schemes, Benjamin/Cummings, Menlo Park, CA, 1984.

[Ba04] R.A. Bailey, Association Schemes: Designed Experiments, Algebra and Combinatorics, Cambridge University Press, Cambridge, 2004

[BCN89] A.E. Brouwer, A.M. Cohen, A. Neumaier, Distance-Regular Graphs, Springer-Verlag, Berlin, 1989

[Z96] P.-H. Zieschang, An algebraic approach to association schemes, Springer-Verlag, Berlin,1996

[Z05] P.-H. Zieschang, Theory of Association Schemes, Springer-Verlag, Berlin, 2005.

Binary relations

Let $R, S \subseteq \Omega^2$ be binary relations. Then

- $S^* := \{(\alpha, \beta) \mid (\beta, \alpha) \in S\}$;
- S is symmetric (antisymmetric) if $S = S^*$ ($S \cap S^* = \emptyset$ resp.);
- $\alpha S := \{\beta \mid (\alpha, \beta) \in S\}, S\alpha := \alpha S^*$;
- $D(S) := \{\alpha \in \Omega \mid \alpha S \neq \emptyset\}, R(S) := D(S^*)$;
- $RS = \{(\alpha, \beta) \mid \alpha R \cap S\beta \neq \emptyset\}$;
- $R^+ = \bigcup_{i=1}^{\infty} R^i$ is the transitive closure of R;
- $1_\Omega := \{(\omega, \omega) \mid \omega \in \Omega\}$

Each permutation $g \in \mathrm{Sym}(\Omega)$ is considered as a binary relation. Thus $\alpha g = \{\alpha^g\}$ and $g^* = g^{-1}$.

Partitions.

- $\mathcal{P} \vdash \Omega$ means that $\mathcal{P}$ is a partition of Ω.
- $\mathcal{P} \sqsubseteq \mathcal{C} \iff \mathcal{C}$ is a refinement of $\mathcal{P}$;
- Lattice operations are denoted as $\mathcal{P} \vee \mathcal{C}$ and $\mathcal{P} \wedge \mathcal{C}$;
- if $\mathcal{P} \vdash \Omega$ then $\mathcal{P}^{\cup}$ denotes the set of all possible unions of elements in $\mathcal{P}$;
- $\mathcal{C} \vdash \Omega^2 \implies \mathcal{C}^* := \{C^* \mid C \in \mathcal{C}\}$;

Graphs.

In what follows graph is a pair $\Gamma = (\Omega, E)$ where Ω is a finite set of vertices and $E \subset \Omega \times \Omega$ is the set of (directed) edges/arcs.

Graphs.

In what follows graph is a pair $\Gamma = (\Omega, E)$ where Ω is a finite set of vertices and $E \subset \Omega \times \Omega$ is the set of (directed) edges/arcs.

Definition.

Graphs $\Gamma_1 = (\Omega_1, E_1)$ and $\Gamma_2 = (\Omega_2, E_2)$ are called isomorphic, $\Gamma_1 \cong \Gamma_2$, if there is a bijection $f : \Omega_1 \to \Omega_2$ such that

$$\forall \alpha_1, \beta_1 \in \Omega_1 : \quad (\alpha_1^f, \beta_1^f) \in E_2 \quad \Leftrightarrow \quad (\alpha_1, \beta_1) \in E_1.$$

Such a bijection is called an isomorphism from Γ_1 to Γ_2; the set of all of them is denoted by $\mathrm{Iso}(\Gamma_1, \Gamma_2)$.The set $\mathrm{Iso}(\Gamma_1, \Gamma_1)$ is known as the automorphism group of Γ_1, notation $\mathrm{Aut}(\Gamma_1)$.

Graphs.

In what follows graph is a pair $\Gamma = (\Omega, E)$ where Ω is a finite set of vertices and $E \subset \Omega \times \Omega$ is the set of (directed) edges/arcs.

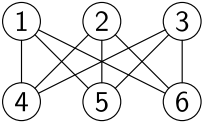
Definition.

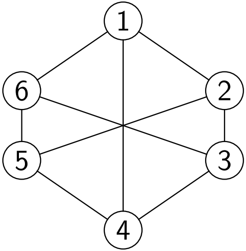
Graphs $\Gamma_1 = (\Omega_1, E_1)$ and $\Gamma_2 = (\Omega_2, E_2)$ are called isomorphic, $\Gamma_1 \cong \Gamma_2$, if there is a bijection $f : \Omega_1 \to \Omega_2$ such that

$$\forall \alpha_1, \beta_1 \in \Omega_1 : \quad (\alpha_1^f, \beta_1^f) \in E_2 \quad \Leftrightarrow \quad (\alpha_1, \beta_1) \in E_1.$$

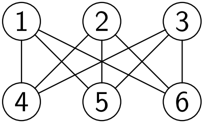
Such a bijection is called an isomorphism from Γ_1 to Γ_2; the set of all of them is denoted by $\mathrm{Iso}(\Gamma_1, \Gamma_2)$.The set $\mathrm{Iso}(\Gamma_1, \Gamma_1)$ is known as the automorphism group of Γ_1, notation $\mathrm{Aut}(\Gamma_1)$.

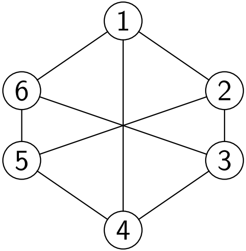
Example





Example

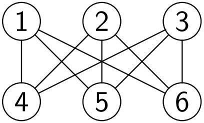


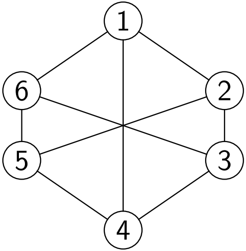


An isomorphism

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 5 & 2 & 4 & 6 \end{pmatrix}$$

Example





An isomorphism

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 5 & 2 & 4 & 6 \end{pmatrix}$$

$$\mathrm{Aut}(\Gamma) = (S_3 \times S_3).S_2.$$

The Graph Isomorphism Problem (ISO).

ISO is to find the computational complexity of the problem:

The Graph Isomorphism Problem (ISO).

ISO is to find the computational complexity of the problem:

ISO(Γ_1, Γ_2): given graphs Γ_1 and Γ_2 test whether or not $\Gamma_1 \cong \Gamma_2$.

The Graph Isomorphism Problem (ISO).

ISO is to find the computational complexity of the problem:

ISO(Γ_1, Γ_2): given graphs Γ_1 and Γ_2 test whether or not $\Gamma_1 \cong \Gamma_2$.

- Given graphs Γ_1 and Γ_2 of order n, and a bijection $f : \Omega_1 \to \Omega_2$ one can test in time $O(n^2)$ whether $f \in \mathrm{Iso}(\Gamma_1, \Gamma_2)$.

The Graph Isomorphism Problem (ISO).

ISO is to find the computational complexity of the problem:

ISO(Γ_1, Γ_2): given graphs Γ_1 and Γ_2 test whether or not $\Gamma_1 \cong \Gamma_2$.

- Given graphs Γ_1 and Γ_2 of order n, and a bijection $f : \Omega_1 \to \Omega_2$ one can test in time $O(n^2)$ whether $f \in \mathrm{Iso}(\Gamma_1, \Gamma_2)$.
- Therefore ISO$\in$**NP**.

The Graph Isomorphism Problem (ISO).

ISO is to find the computational complexity of the problem:

ISO(Γ_1, Γ_2): given graphs Γ_1 and Γ_2 test whether or not $\Gamma_1 \cong \Gamma_2$.

- Given graphs Γ_1 and Γ_2 of order n, and a bijection $f : \Omega_1 \to \Omega_2$ one can test in time $O(n^2)$ whether $f \in \mathrm{Iso}(\Gamma_1, \Gamma_2)$.
- Therefore ISO$\in$**NP**.
- An exhaustive search of all the possible bijections runs in exponential time $O(n!)$.

The Graph Isomorphism Problem (ISO).

ISO is to find the computational complexity of the problem:

ISO(Γ_1, Γ_2): given graphs Γ_1 and Γ_2 test whether or not $\Gamma_1 \cong \Gamma_2$.

- Given graphs Γ_1 and Γ_2 of order n, and a bijection $f : \Omega_1 \to \Omega_2$ one can test in time $O(n^2)$ whether $f \in \mathrm{Iso}(\Gamma_1, \Gamma_2)$.
- Therefore ISO$\in$**NP**.
- An exhaustive search of all the possible bijections runs in exponential time $O(n!)$.
- At present it is not known whether ISO$\in$**P**.

The Graph Isomorphism Problem (ISO).

ISO is to find the computational complexity of the problem:

ISO(Γ_1, Γ_2): given graphs Γ_1 and Γ_2 test whether or not $\Gamma_1 \cong \Gamma_2$.

- Given graphs Γ_1 and Γ_2 of order n, and a bijection $f : \Omega_1 \to \Omega_2$ one can test in time $O(n^2)$ whether $f \in \mathrm{Iso}(\Gamma_1, \Gamma_2)$.
- Therefore ISO$\in$**NP**.
- An exhaustive search of all the possible bijections runs in exponential time $O(n!)$.
- At present it is not known whether ISO$\in$**P**.

The proof of the time bound of the best algorithm (up to now) for the ISO depends on the Classification of Finite Simple Groups.

The Graph Isomorphism Problem (ISO).

ISO is to find the computational complexity of the problem:

ISO(Γ_1, Γ_2): given graphs Γ_1 and Γ_2 test whether or not $\Gamma_1 \cong \Gamma_2$.

- Given graphs Γ_1 and Γ_2 of order n, and a bijection $f : \Omega_1 \to \Omega_2$ one can test in time $O(n^2)$ whether $f \in \mathrm{Iso}(\Gamma_1, \Gamma_2)$.
- Therefore ISO$\in$**NP**.
- An exhaustive search of all the possible bijections runs in exponential time $O(n!)$.
- At present it is not known whether ISO$\in$**P**.

The proof of the time bound of the best algorithm (up to now) for the ISO depends on the Classification of Finite Simple Groups.

Theorem (L.Babai, E.Luks and W.Kantor, 1984).

The isomorphism of n-vertex graphs can be tested in time $\exp(O(\sqrt{n \log n}))$.

Some problems equivalent to the ISO (R.Mathon,1979).

The following problems are equivalent to the ISO:

- ICOUNT: given Γ and Γ' find $|\mathrm{Iso}(\Gamma, \Gamma')|$,

Some problems equivalent to the ISO (R.Mathon,1979).

The following problems are equivalent to the ISO:

- ICOUNT: given Γ and Γ' find $|\mathrm{Iso}(\Gamma, \Gamma')|$,
- ACOUNT: given Γ find $|\mathrm{Aut}(\Gamma)|$,

Some problems equivalent to the ISO (R.Mathon,1979).

The following problems are equivalent to the ISO:

- ICOUNT: given Γ and Γ' find $|\mathrm{Iso}(\Gamma, \Gamma')|$,
- ACOUNT: given Γ find $|\mathrm{Aut}(\Gamma)|$,
- AGEN: given Γ find generators of the group $\mathrm{Aut}(\Gamma)$,

Some problems equivalent to the ISO (R.Mathon,1979).

The following problems are equivalent to the ISO:

- ICOUNT: given Γ and Γ' find $|\mathrm{Iso}(\Gamma, \Gamma')|$,
- ACOUNT: given Γ find $|\mathrm{Aut}(\Gamma)|$,
- AGEN: given Γ find generators of the group $\mathrm{Aut}(\Gamma)$,
- APART: given Γ find $\mathrm{Orb}(\mathrm{Aut}(\Gamma))$.

Some problems equivalent to the ISO (R.Mathon,1979).

The following problems are equivalent to the ISO:

- ICOUNT: given Γ and Γ' find $|\text{Iso}(\Gamma, \Gamma')|$,
- ACOUNT: given Γ find $|\text{Aut}(\Gamma)|$,
- AGEN: given Γ find generators of the group $\text{Aut}(\Gamma)$,
- APART: given Γ find $\text{Orb}(\text{Aut}(\Gamma))$.

Isomorphism problem for colored graphs.

Definition.

A triple (Ω, Y, c) where $c : \Omega^2 \to Y$ is a surjection, is called a colored graph with the coloring function c and color classes $c^{-1}(y)$, $y \in Y$. Each colored graph determines a partition $\mathcal{C} := \{c^{-1}(y) \,|\, y \in Y\}$ of Ω^2.

Two colored graphs (Ω, Y, c) and (Δ, Z, d) are isomorphic iff there exist bijections $f : \Omega \to \Delta, \phi : Y \to Z$ s.t.

$$d(\alpha^f, \beta^f) = c(\alpha, \beta)^\phi.$$

Isomorphism problem for colored graphs.

Definition.

A triple (Ω, Y, c) where $c : \Omega^2 \to Y$ is a surjection, is called a colored graph with the coloring function c and color classes $c^{-1}(y)$, $y \in Y$. Each colored graph determines a partition $\mathcal{C} := \{c^{-1}(y) \mid y \in Y\}$ of Ω^2.

Two colored graphs (Ω, Y, c) and (Δ, Z, d) are isomorphic iff there exist bijections $f : \Omega \to \Delta, \phi : Y \to Z$ s.t.

$$d(\alpha^f, \beta^f) = c(\alpha, \beta)^\phi.$$

Notice that ϕ is uniquely determined by f. For this reason we define $f^* := \phi$.

Isomorphism problem for colored graphs.

We also set $\mathrm{Iso}(\Omega, Y, c)$ for the group of all isomorphisms from (Ω, Y, c) to itself and $\mathrm{Aut}(\Omega, Y, c)$ for the normal subgroup of $\mathrm{Iso}(\Omega, Y, c)$ which does not interchanges the colors (that is $f^* = 1_Y$).

Proposition

Let (Ω, Y, c) be a colored graph and $\mathcal{C} := \{c^{-1}(y) \,|\, y \in Y\}$ be the corresponding partition. Then

$$\mathrm{Iso}(\Omega, Y, c) = \{g \in \mathrm{Sym}(\Omega) \,|\, \mathcal{C}^g = \mathcal{C}\},$$
$$\mathrm{Aut}(\Omega, Y, c) = \{g \in \mathrm{Sym}(\Omega) \,|\, \forall_{C \in \mathcal{C}} C^g = C\}.$$

Theorem.

Isomorphism problem for colored graphs is polynomially equivalent to ISO.

Cayley Graphs and their Isomorphisms.

A Cayley graph over a finite group H defined by a connection set $S \subseteq H$ has H as a set of nodes and arc set

$$\mathrm{Cay}(H, S) := \{(x, y) \mid xy^{-1} \in S\}$$

.

A circulant graph is a Cayley graph over a cyclic group.

Definition

Two Cayley graphs $\mathrm{Cay}(H, S)$ and $\mathrm{Cay}(K, T)$ are Cayley isomorphic if there exists a group isomorphism $f : H \to K$ which is a graph isomoprhism too, that is

$$\mathrm{Cay}(H, S)^f = \mathrm{Cay}(K, T) \iff S^f = T.$$

Cayley representations of graphs.

An automorphism of a Cayley graph $\mathrm{Cay}(H, S)$ contains a regular subgroup H_R consisting of right translations $h_R, h \in H$:

$$x^{h_R} = xh, x \in H.$$

Theorem (Sabidussi)

A graph $\Gamma = (\Omega, E)$ is isomorphic to a Cayley graph over a group H iff $\mathrm{Aut}(\Gamma)$ contains a regular subgroup isomorphic to H.

Isomorphism problems for Cayley graphs.

Given $\Gamma = \mathrm{Cay}(H, S)$ and $\Gamma' = \mathrm{Cay}(H, S')$:

- IMAP: find $f \in \mathrm{Iso}(\Gamma, \Gamma')$ (if it exists),
- ICOUNT: find $|\mathrm{Iso}(\Gamma, \Gamma')|$,
- ACOUNT:find $|\mathrm{Aut}(\Gamma)|$,
- AGEN: find generators of the group $\mathrm{Aut}(\Gamma)$,
- CGR: given a graph Θ find whether it's a Cayley graph over a group H.

Isomorphism problem for finite groups.

Construction. Let K be a finite group.

Define a graph $\Gamma(K)$ with vertex set $K \times K$ and edges:

$$(a, b) \sim (c, d) \iff a = c \vee b = d \vee ab = cd.$$

Isomorphism problem for finite groups.

Construction. Let K be a finite group.

Define a graph $\Gamma(K)$ with vertex set $K \times K$ and edges:

$$(a,b) \sim (c,d) \iff a = c \vee b = d \vee ab = cd.$$

Theorem $K_1 \cong K_2 \iff \Gamma(K_1) \cong \Gamma(K_2)$.

Isomorphism problem for finite groups.

Construction. Let K be a finite group.

Define a graph $\Gamma(K)$ with vertex set $K \times K$ and edges:

$$(a,b) \sim (c,d) \iff a = c \vee b = d \vee ab = cd.$$

Theorem $K_1 \cong K_2 \iff \Gamma(K_1) \cong \Gamma(K_2)$.

Exercise. Prove that $\Gamma(K)$ is a Cayley graph over $K \times K$.

Exercise. Prove that $\Gamma(\mathbb{Z}_4) \ncong \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)$.

$\mathbb{Z}_4 \to$

0	1	2	3
1	2	3	0
2	3	0	1
3	0	1	2

$\mathbb{Z}_2 \times \mathbb{Z}_2 \to$

0	1	2	3
1	0	3	2
2	3	0	1
3	2	1	0

Isomorphism problem for finite groups.

Construction. Let K be a finite group.

Define a graph $\Gamma(K)$ with vertex set $K \times K$ and edges:

$$(a, b) \sim (c, d) \iff a = c \vee b = d \vee ab = cd.$$

Theorem $K_1 \cong K_2 \iff \Gamma(K_1) \cong \Gamma(K_2)$.

Exercise. Prove that $\Gamma(K)$ is a Cayley graph over $K \times K$.

Exercise. Prove that $\Gamma(\mathbb{Z}_4) \not\cong \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)$.

$\mathbb{Z}_4 \to$

0	1	2	3
1	2	3	0
2	3	0	1
3	0	1	2

$\mathbb{Z}_2 \times \mathbb{Z}_2 \to$

0	1	2	3
1	0	3	2
2	3	0	1
3	2	1	0

The isomorphism of groups of order n can be tested in time $n^{O(\log n)}$.

Naive vertex classification.

Naive vertex classification.

Vertex partition by valences.

Denote by $d_\Gamma(\alpha)$ the valency of the vertex α in the graph Γ;

Naive vertex classification.

Vertex partition by valences.

Denote by $d_\Gamma(\alpha)$ the valency of the vertex α in the graph Γ;the valency of α in a color class is denoted by $d_\Gamma(\alpha, C)$.

Naive vertex classification.

Vertex partition by valences.

Denote by $d_\Gamma(\alpha)$ the valency of the vertex α in the graph Γ;the valency of α in a color class is denoted by $d_\Gamma(\alpha, C)$.

- To find $\mathrm{Orb}(\mathrm{Aut}(\Gamma))$ put vertices α and β in the same class iff $d_\Gamma(\alpha) = d_\Gamma(\beta)$.

Naive vertex classification.

Vertex partition by valences.

Denote by $d_\Gamma(\alpha)$ the valency of the vertex α in the graph Γ;the valency of α in a color class is denoted by $d_\Gamma(\alpha, C)$.

- To find $\mathrm{Orb}(\mathrm{Aut}(\Gamma))$ put vertices α and β in the same class iff $d_\Gamma(\alpha) = d_\Gamma(\beta)$.
- Iteratively, put vertices α and β in the same class iff $d_\Gamma(\alpha, C) = d_\Gamma(\beta, C)$ for all color classes C.

Naive vertex classification.

Vertex partition by valences.

Denote by $d_\Gamma(\alpha)$ the valency of the vertex α in the graph Γ;the valency of α in a color class is denoted by $d_\Gamma(\alpha, C)$.

- To find $\mathrm{Orb}(\mathrm{Aut}(\Gamma))$ put vertices α and β in the same class iff $d_\Gamma(\alpha) = d_\Gamma(\beta)$.
- Iteratively, put vertices α and β in the same class iff $d_\Gamma(\alpha, C) = d_\Gamma(\beta, C)$ for all color classes C.

Comments.

- The algorithm correctly finds $\mathrm{Orb}(\mathrm{Aut}(\Gamma))$ for the class of trees (G.Tinhofer, 1985), for almost all graphs (L.Babai, P.Erdös, S.Selkow, 1980).

Naive vertex classification.

Vertex partition by valences.

Denote by $d_\Gamma(\alpha)$ the valency of the vertex α in the graph Γ;the valency of α in a color class is denoted by $d_\Gamma(\alpha, C)$.

- To find Orb(Aut(Γ)) put vertices α and β in the same class iff $d_\Gamma(\alpha) = d_\Gamma(\beta)$.
- Iteratively, put vertices α and β in the same class iff $d_\Gamma(\alpha, C) = d_\Gamma(\beta, C)$ for all color classes C.

Comments.

- The algorithm correctly finds Orb(Aut(Γ)) for the class of trees (G.Tinhofer, 1985), for almost all graphs (L.Babai, P.Erdös, S.Selkow, 1980).
- The algorithm fails when Γ is a regular graphs and the group Aut(Γ) is intransitive.

Naive vertex classification.

Vertex partition by valences.

Denote by $d_\Gamma(\alpha)$ the valency of the vertex α in the graph Γ;the valency of α in a color class is denoted by $d_\Gamma(\alpha, C)$.

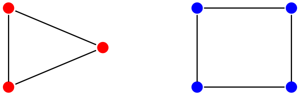
- To find Orb(Aut(Γ)) put vertices α and β in the same class iff $d_\Gamma(\alpha) = d_\Gamma(\beta)$.
- Iteratively, put vertices α and β in the same class iff $d_\Gamma(\alpha, C) = d_\Gamma(\beta, C)$ for all color classes C.

Comments.

- The algorithm correctly finds Orb(Aut(Γ)) for the class of trees (G.Tinhofer, 1985), for almost all graphs (L.Babai, P.Erdös, S.Selkow, 1980).
- The algorithm fails when Γ is a regular graphs and the group Aut(Γ) is intransitive.

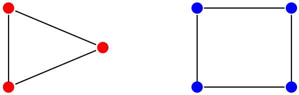
The Weisfeiler-Leman algorithm, 1968.

No automorphism moves red points to blue ones.



The Weisfeiler-Leman algorithm, 1968.

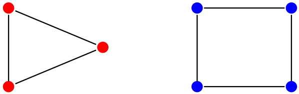
No automorphism moves red points to blue ones.



To distinguish vertices we need to color edges of Γ.

The Weisfeiler-Leman algorithm, 1968.

No automorphism moves red points to blue ones.

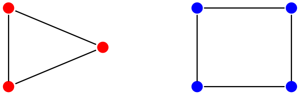


To distinguish vertices we need to color edges of Γ.

Algorithm. Set $\mathcal{C} = \{1_\Omega\} \cup \{E\} \cup \{(\Omega \times \Omega) \setminus E\}$.

The Weisfeiler-Leman algorithm, 1968.

No automorphism moves red points to blue ones.



To distinguish vertices we need to color edges of Γ.

Algorithm. Set $\mathcal{C} = \{1_\Omega\} \cup \{E\} \cup \{(\Omega \times \Omega) \setminus E\}$.

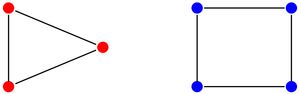
- For all $(\alpha, \beta) \in \Omega \times \Omega$ and $R, S \in \mathcal{C}$ find the number

$$c(\alpha, \beta; R, S) = |\alpha R \cap S\beta|.$$

- Build a new partition $\text{Ы}(\mathcal{C})$ by putting (α, β) and (α', β') to the same class of $\text{Ы}(\mathcal{C})$ if $|\alpha R \cap S\beta| = |\alpha' R \cap S\beta'|$ for all $R, S \in \mathcal{C}$.

The Weisfeiler-Leman algorithm, 1968.

No automorphism moves red points to blue ones.



To distinguish vertices we need to color edges of Γ.

Algorithm. Set $\mathcal{C} = \{1_\Omega\} \cup \{E\} \cup \{(\Omega \times \Omega) \setminus E\}$.

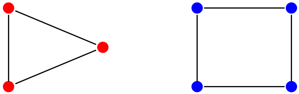
- For all $(\alpha, \beta) \in \Omega \times \Omega$ and $R, S \in \mathcal{C}$ find the number

$$c(\alpha, \beta; R, S) = |\alpha R \cap S\beta|.$$

- Build a new partition $\text{Ы}(\mathcal{C})$ by putting (α, β) and (α', β') to the same class of $\text{Ы}(\mathcal{C})$ if $|\alpha R \cap S\beta| = |\alpha' R \cap S\beta'|$ for all $R, S \in \mathcal{C}$.
- Repeat the procedure till $|\mathcal{C}|$ stops to increase.

The Weisfeiler-Leman algorithm, 1968.

No automorphism moves red points to blue ones.

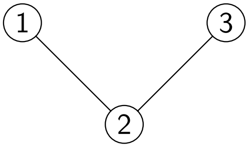


To distinguish vertices we need to color edges of Γ.

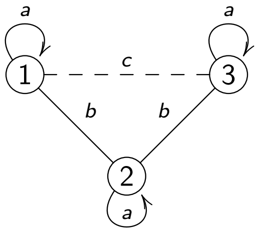
Algorithm. Set $\mathcal{C} = \{1_\Omega\} \cup \{E\} \cup \{(\Omega \times \Omega) \setminus E\}$.

- For all $(\alpha, \beta) \in \Omega \times \Omega$ and $R, S \in \mathcal{C}$ find the number
$$c(\alpha, \beta; R, S) = |\alpha R \cap S\beta|.$$
- Build a new partition Ы$(\mathcal{C})$ by putting (α, β) and (α', β') to the same class of Ы$(\mathcal{C})$ if $|\alpha R \cap S\beta| = |\alpha' R \cap S\beta'|$ for all $R, S \in \mathcal{C}$.
- Repeat the procedure till $|\mathcal{C}|$ stops to increase.

The WL-algorithm. Very small example



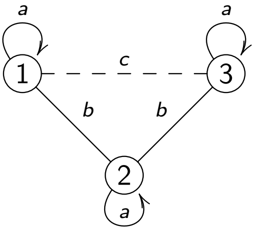
Initial coloring



Adjacency matrix

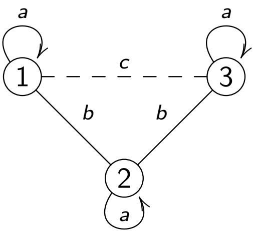
$$A = \begin{pmatrix} a & b & c \\ b & a & b \\ c & b & a \end{pmatrix}.$$

First iteration



$$A^2 = \begin{pmatrix} a^2 + b^2 + c^2 & ab + ba + cb & ac + b^2 + ca \\ ba + ab + bc & 2b^2 + a^2 & bc + ab + ba \\ ca + b^2 + ac & cb + ba + ab & c^2 + b^2 + a^2 \end{pmatrix}.$$

First iteration



$$A^2 = \begin{pmatrix} a^2 + b^2 + c^2 & ab + ba + cb & ac + b^2 + ca \\ bc + ab + ba & 2b^2 + a^2 & bc + ab + ba \\ ac + b^2 + ca & ab + ba + cb & c^2 + b^2 + a^2 \end{pmatrix}.$$

Second iteration

New matrix A

$$A = \begin{pmatrix} d & e & f \\ g & h & g \\ f & e & d \end{pmatrix}$$

Second iteration

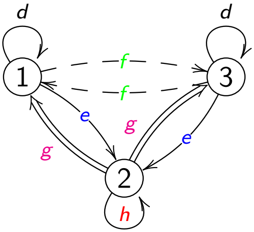
New matrix A

$$A = \begin{pmatrix} d & e & f \\ g & h & g \\ f & e & d \end{pmatrix}$$

Second iteration

New matrix A

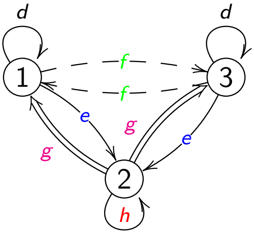
$$A = \begin{pmatrix} d & e & f \\ g & h & g \\ f & e & d \end{pmatrix}$$



Second iteration

New matrix A

$$A = \begin{pmatrix} d & e & f \\ g & h & g \\ f & e & d \end{pmatrix}$$



The matrix A is stable, that is A^2 produces the same coloring as A does.

WL-refinement (operation Ы)

Properties

WL-refinement (operation Ы)

Properties

- $\mathcal{C} \sqsubseteq \mathcal{S} \implies \text{Ы}(\mathcal{C}) \sqsubseteq \text{Ы}(\mathcal{S})$;

WL-refinement (operation Ы)

Properties

- $\mathcal{C} \sqsubseteq \mathcal{S} \implies \text{Ы}(\mathcal{C}) \sqsubseteq \text{Ы}(\mathcal{S})$;
- $\mathcal{C}^* = \mathcal{C} \implies \text{Ы}(\mathcal{C})^* = \text{Ы}(\mathcal{C})$;

WL-refinement (operation Ы)

Properties

- $\mathcal{C} \sqsubseteq \mathcal{S} \implies \text{Ы}(\mathcal{C}) \sqsubseteq \text{Ы}(\mathcal{S})$;
- $\mathcal{C}^* = \mathcal{C} \implies \text{Ы}(\mathcal{C})^* = \text{Ы}(\mathcal{C})$;
- $1_\Omega \in \mathcal{C}^\cup \implies \mathcal{C} \sqsubseteq \text{Ы}(\mathcal{C})$;

WL-refinement (operation Ы)

Properties

- $\mathcal{C} \sqsubseteq \mathcal{S} \implies Ы(\mathcal{C}) \sqsubseteq Ы(\mathcal{S})$;
- $\mathcal{C}^* = \mathcal{C} \implies Ы(\mathcal{C})^* = Ы(\mathcal{C})$;
- $1_\Omega \in \mathcal{C}^\cup \implies \mathcal{C} \sqsubseteq Ы(\mathcal{C})$;

Proposition

Let $f : \Omega \to \Delta$ be a bijection that maps a partition $\mathcal{C}$ of Ω^2 onto a paritition $\mathcal{T}$ of Δ^2 (i.e. $\mathcal{C}^f = \mathcal{T}$). Then $Ы(\mathcal{C})^f = Ы(\mathcal{T})$.

WL-refinement (operation Ы)

Properties

- $\mathcal{C} \sqsubseteq \mathcal{S} \implies \text{Ы}(\mathcal{C}) \sqsubseteq \text{Ы}(\mathcal{S})$;
- $\mathcal{C}^* = \mathcal{C} \implies \text{Ы}(\mathcal{C})^* = \text{Ы}(\mathcal{C})$;
- $1_\Omega \in \mathcal{C}^\cup \implies \mathcal{C} \sqsubseteq \text{Ы}(\mathcal{C})$;

Proposition

Let $f : \Omega \to \Delta$ be a bijection that maps a partition $\mathcal{C}$ of Ω^2 onto a paritition $\mathcal{T}$ of Δ^2 (i.e. $\mathcal{C}^f = \mathcal{T}$). Then $\text{Ы}(\mathcal{C})^f = \text{Ы}(\mathcal{T})$.

Given an ordered partition $\vec{\mathcal{C}} = (S_1, ..., S_m)$ of Ω^2 the WL-algorithm produces a unique (canonical) ordering of the refinement $\text{Ы}(\mathcal{C})$ (denoted as $\text{Ы}(\vec{\mathcal{C}})$) with the following property:

$$\vec{\mathcal{C}}^f = \vec{\mathcal{T}} \implies \text{Ы}(\vec{\mathcal{C}})^f = \text{Ы}(\vec{\mathcal{T}})$$

Coherent configurations (D. Higman, 1970).

The output partition of the Weisfeiler-Leman algorithm is a coherent configuration, i.e. a pair $\mathcal{X} = (\Omega, \mathcal{C})$ such that:

Coherent configurations (D. Higman, 1970).

The output partition of the Weisfeiler-Leman algorithm is a coherent configuration, i.e. a pair $\mathcal{X} = (\Omega, \mathcal{C})$ such that:

- $\mathcal{C}$ is a partition of $\Omega \times \Omega$,

Coherent configurations (D. Higman, 1970).

The output partition of the Weisfeiler-Leman algorithm is a coherent configuration, i.e. a pair $\mathcal{X} = (\Omega, \mathcal{C})$ such that:

- $\mathcal{C}$ is a partition of $\Omega \times \Omega$,
- $1_\Omega \in \mathcal{C}^\cup$,

Coherent configurations (D. Higman, 1970).

The output partition of the Weisfeiler-Leman algorithm is a coherent configuration, i.e. a pair $\mathcal{X} = (\Omega, \mathcal{C})$ such that:

- $\mathcal{C}$ is a partition of $\Omega \times \Omega$,
- $1_\Omega \in \mathcal{C}^\cup$,
- $\mathcal{C}^* = \mathcal{C}$,

Coherent configurations (D. Higman, 1970).

The output partition of the Weisfeiler-Leman algorithm is a coherent configuration, i.e. a pair $\mathcal{X} = (\Omega, \mathcal{C})$ such that:

- $\mathcal{C}$ is a partition of $\Omega \times \Omega$,
- $1_\Omega \in \mathcal{C}^\cup$,
- $\mathcal{C}^* = \mathcal{C}$,
- $\mathcal{C} = \text{Ы}(\mathcal{C})$,

Coherent configurations (D. Higman, 1970).

The output partition of the Weisfeiler-Leman algorithm is a coherent configuration, i.e. a pair $\mathcal{X} = (\Omega, \mathcal{C})$ such that:

- $\mathcal{C}$ is a partition of $\Omega \times \Omega$,
- $1_\Omega \in \mathcal{C}^\cup$,
- $\mathcal{C}^* = \mathcal{C}$,
- $\mathcal{C} =$ Ы$(\mathcal{C})$,that is for all $R, S, T \in \mathcal{C}$ the intersection number $c_{RS}^T = |\alpha R \cap S\beta|$ does not depend on the choice of $(\alpha, \beta) \in T$.

Coherent configurations (D. Higman, 1970).

The output partition of the Weisfeiler-Leman algorithm is a coherent configuration, i.e. a pair $\mathcal{X} = (\Omega, \mathcal{C})$ such that:

- $\mathcal{C}$ is a partition of $\Omega \times \Omega$,
- $1_\Omega \in \mathcal{C}^\cup$,
- $\mathcal{C}^* = \mathcal{C}$,
- $\mathcal{C} =$ Ы$(\mathcal{C})$,that is for all $R, S, T \in \mathcal{C}$ the intersection number $c_{RS}^T = |\alpha R \cap S\beta|$ does not depend on the choice of $(\alpha, \beta) \in T$.
- the degree and rank of $\mathcal{X}$ are the numbers $|\Omega|$ and $|\mathcal{C}|$,

Coherent configurations (D. Higman, 1970).

The output partition of the Weisfeiler-Leman algorithm is a coherent configuration, i.e. a pair $\mathcal{X} = (\Omega, \mathcal{C})$ such that:

- $\mathcal{C}$ is a partition of $\Omega \times \Omega$,
- $1_\Omega \in \mathcal{C}^\cup$,
- $\mathcal{C}^* = \mathcal{C}$,
- $\mathcal{C} =$ Ы$(\mathcal{C})$,that is for all $R, S, T \in \mathcal{C}$ the intersection number $c_{RS}^T = |\alpha R \cap S\beta|$ does not depend on the choice of $(\alpha, \beta) \in T$.
- the degree and rank of $\mathcal{X}$ are the numbers $|\Omega|$ and $|\mathcal{C}|$,
- the basic relations and relations of $\mathcal{X}$ are the relations of $\mathcal{C}$ and of $\mathcal{C}^\cup$.

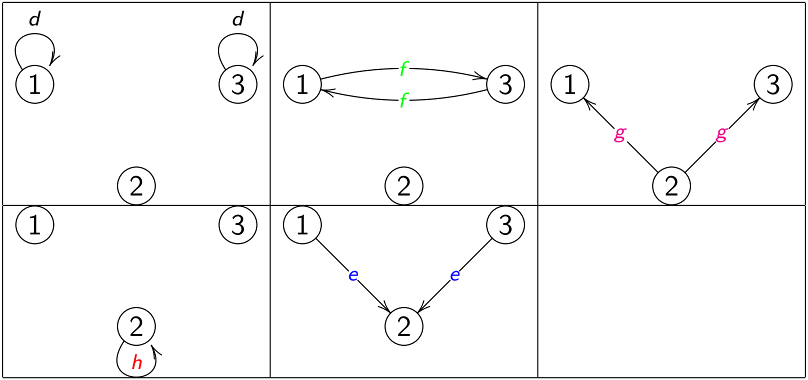
Coherent configurations (D. Higman, 1970).

The output partition of the Weisfeiler-Leman algorithm is a coherent configuration, i.e. a pair $\mathcal{X} = (\Omega, \mathcal{C})$ such that:

- $\mathcal{C}$ is a partition of $\Omega \times \Omega$,
- $1_\Omega \in \mathcal{C}^\cup$,
- $\mathcal{C}^* = \mathcal{C}$,
- $\mathcal{C} = \text{Ы}(\mathcal{C})$,that is for all $R, S, T \in \mathcal{C}$ the intersection number $c_{RS}^T = |\alpha R \cap S\beta|$ does not depend on the choice of $(\alpha, \beta) \in T$.
- the degree and rank of $\mathcal{X}$ are the numbers $|\Omega|$ and $|\mathcal{C}|$,
- the basic relations and relations of $\mathcal{X}$ are the relations of $\mathcal{C}$ and of $\mathcal{C}^\cup$.

The configuration $\mathcal{X}$ is homogeneous (or association scheme, or scheme), if $1_\Omega \in \mathcal{C}$.

Coherent configurations: a concrete example.



Coherent configurations. Fibers and relations.

A fiber of $\mathcal{X}$ is a set $\Delta \subset \Omega$ such that $1_\Delta \in \mathcal{C}$; the set of all fibers is denoted by $\Phi = \Phi(\mathcal{X})$.

Coherent configurations. Fibers and relations.

A fiber of $\mathcal{X}$ is a set $\Delta \subset \Omega$ such that $1_\Delta \in \mathcal{C}$; the set of all fibers is denoted by $\Phi = \Phi(\mathcal{X})$.Thus $\mathcal{X}$ is a scheme iff $|\Phi| = 1$.

Coherent configurations. Fibers and relations.

A fiber of $\mathcal{X}$ is a set $\Delta \subset \Omega$ such that $1_\Delta \in \mathcal{C}$; the set of all fibers is denoted by $\Phi = \Phi(\mathcal{X})$.Thus $\mathcal{X}$ is a scheme iff $|\Phi| = 1$.

Proposition. The following statements hold:

Coherent configurations. Fibers and relations.

A fiber of $\mathcal{X}$ is a set $\Delta \subset \Omega$ such that $1_\Delta \in \mathcal{C}$; the set of all fibers is denoted by $\Phi = \Phi(\mathcal{X})$.Thus $\mathcal{X}$ is a scheme iff $|\Phi| = 1$.

Proposition. The following statements hold:

- $\Omega = \bigcup_{\Delta \in \Phi} \Delta$,

Coherent configurations. Fibers and relations.

A fiber of $\mathcal{X}$ is a set $\Delta \subset \Omega$ such that $1_\Delta \in \mathcal{C}$; the set of all fibers is denoted by $\Phi = \Phi(\mathcal{X})$.Thus $\mathcal{X}$ is a scheme iff $|\Phi| = 1$.

Proposition. The following statements hold:

- $\Omega = \bigcup_{\Delta \in \Phi} \Delta$,
- for any $S \in \mathcal{C}$ the sets $D(S)$ and $R(S)$ are fibres of $\mathcal{X}$,

Coherent configurations. Fibers and relations.

A fiber of $\mathcal{X}$ is a set $\Delta \subset \Omega$ such that $1_\Delta \in \mathcal{C}$; the set of all fibers is denoted by $\Phi = \Phi(\mathcal{X})$.Thus $\mathcal{X}$ is a scheme iff $|\Phi| = 1$.

Proposition. The following statements hold:

- $\Omega = \bigcup_{\Delta \in \Phi} \Delta$,
- for any $S \in \mathcal{C}$ the sets $D(S)$ and $R(S)$ are fibres of $\mathcal{X}$,
- for any $S \in \mathcal{C}$ and $\alpha \in D(S)$ we have $|\alpha S| = c_{SS^*}^T$ where $T = 1_{D(S)}$.

Coherent configurations. Fibers and relations.

A fiber of $\mathcal{X}$ is a set $\Delta \subset \Omega$ such that $1_\Delta \in \mathcal{C}$; the set of all fibers is denoted by $\Phi = \Phi(\mathcal{X})$.Thus $\mathcal{X}$ is a scheme iff $|\Phi| = 1$.

Proposition. The following statements hold:

- $\Omega = \bigcup_{\Delta \in \Phi} \Delta$,
- for any $S \in \mathcal{C}$ the sets $D(S)$ and $R(S)$ are fibres of $\mathcal{X}$,
- for any $S \in \mathcal{C}$ and $\alpha \in D(S)$ we have $|\alpha S| = c_{SS^*}^T$ where $T = 1_{D(S)}$.
- for any fiber $\Delta \in \Phi$ the set of relations $\mathcal{C}_\Delta := \{C \in \mathcal{C} \mid D(C) = \Delta, R(C) = \Delta\}$ form a homogeneous co.co. on Δ, called a homogeneous constituent of $\mathcal{C}$.

The number $n_S = c_{SS^*}^T$ is called the valency of S.

Properties of coherent configurations.

Proposition

Let $\mathcal{X} = (\Omega, \mathcal{C})$ be a co.co. Then

- the set $\mathcal{C}^{\cup}$ is closed w.r.t. boolean operations;
- $1_{\Omega}, \Omega^2 \in \mathcal{C}^{\cup}$;
- $(\mathcal{C}^{\cup})^* = \mathcal{C}^{\cup}$;
- $\mathcal{C}^{\cup}$ is closed w.r.t. relational product;

Isomorphisms between coherent configurations

Definition

Two coherent configuration $\mathcal{X} = (\Omega, \mathcal{C})$ and $\mathcal{X}' = (\Omega', \mathcal{C}')$ are called (combinatorially) isomorphic iff there exist bijections $f : \Omega \to \Omega', \phi : \mathcal{C} \to \mathcal{C}'$ such that

$$\forall_{\alpha,\beta\in\Omega} \;\; (\alpha, \beta) \in C \iff (\alpha^f, \beta^f) \in C^\phi.$$

The set of all isomorphisms between $\mathcal{X}$ and $\mathcal{X}'$ is denoted as $\mathrm{Iso}(\mathcal{X}, \mathcal{X}')$. Notice that ϕ is uniquely determined by f.

In what follows we set $\mathrm{Iso}(\mathcal{X}) := \mathrm{Iso}(\mathcal{X}, \mathcal{X})$. We call the elements of this group colored automorphisms of the configuration.

Coherent configurations generated by a graph.

The mapping $(f, \phi) \mapsto \phi$ is an group homomorphism from $\mathrm{Iso}(\mathcal{X})$ into $\mathrm{Sym}(\mathcal{C})$. The kernel of this homomorphism denoted as $\mathrm{Aut}(\mathcal{X})$ is called the the automorphism group of $\mathcal{X}$:

$$\mathrm{Aut}(\mathcal{X}) = \{f \in \mathrm{Sym}(\Omega) : \ S^f = S \text{ for all } S \in \mathcal{C}\}$$

Theorem

Let $\langle\langle\Gamma\rangle\rangle$ be the WL-closure of a graph $\Gamma = (\Omega, E)$ obtained by applying WL-algorithm to Γ. Then

- $E \in \langle\langle\Gamma\rangle\rangle^{\cup}$;
- $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\langle\langle\Gamma\rangle\rangle)$.

Coherent configurations generated by a graph.

The mapping $(f, \phi) \mapsto \phi$ is an group homomorphism from $\mathrm{Iso}(\mathcal{X})$ into $\mathrm{Sym}(\mathcal{C})$. The kernel of this homomorphism denoted as $\mathrm{Aut}(\mathcal{X})$ is called the the automorphism group of $\mathcal{X}$:

$$\mathrm{Aut}(\mathcal{X}) = \{f \in \mathrm{Sym}(\Omega) : \ S^f = S \text{ for all } S \in \mathcal{C}\}$$

Theorem

Let $\langle\langle\Gamma\rangle\rangle$ be the WL-closure of a graph $\Gamma = (\Omega, E)$ obtained by applying WL-algorithm to Γ. Then

- $E \in \langle\langle\Gamma\rangle\rangle^{\cup}$;
- $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\langle\langle\Gamma\rangle\rangle)$.

Examples. Strongly regular graphs.

Definition

A graph $\Gamma = (\Omega, E)$ is called strongly regular if its WL-closure has rank three. In other words, WL-algorithm stops at the first iteration and $\langle\langle\Gamma\rangle\rangle = \{1_\Omega, E, E^c\}$.

Examples. Strongly regular graphs.

Definition

A graph $\Gamma = (\Omega, E)$ is called strongly regular if its WL-closure has rank three. In other words, WL-algorithm stops at the first iteration and $\langle\langle\Gamma\rangle\rangle = \{1_\Omega, E, E^c\}$.

Proposition

A graph $\Gamma = (\Omega, E)$ is strongly regular if and only if there exists non-negative integers k, λ, μ such that

1. Γ is k-regular,
2. any pair of points connected by an edge have λ common neighbours,
3. any pair of points not connected by an edge have μ common neighbours

Examples. Permutation groups.

Let $G \leq \mathrm{Sym}(\Omega)$ be a permutation group. It acts on $\Omega \times \Omega$:

$$(\alpha, \beta)^g := (\alpha^g, \beta^g), \qquad \alpha, \beta \in \Omega,\ g \in G.$$

Examples. Permutation groups.

Let $G \leq \mathrm{Sym}(\Omega)$ be a permutation group. It acts on $\Omega \times \Omega$:

$$(\alpha, \beta)^g := (\alpha^g, \beta^g), \qquad \alpha, \beta \in \Omega, \; g \in G.$$

Set $\mathrm{Inv}(G) := (\Omega, \mathcal{C})$ where $\mathcal{C} := \mathrm{Orb}(G, \Omega \times \Omega)$. Then

Examples. Permutation groups.

Let $G \leq \mathrm{Sym}(\Omega)$ be a permutation group. It acts on $\Omega \times \Omega$:

$$(\alpha, \beta)^g := (\alpha^g, \beta^g), \qquad \alpha, \beta \in \Omega,\ g \in G.$$

Set $\mathrm{Inv}(G) := (\Omega, \mathcal{C})$ where $\mathcal{C} := \mathrm{Orb}(G, \Omega \times \Omega)$. Then

1. $\mathrm{Inv}(G)$ is a coherent configuration (of G),

Examples. Permutation groups.

Let $G \leq \mathrm{Sym}(\Omega)$ be a permutation group. It acts on $\Omega \times \Omega$:

$$(\alpha, \beta)^g := (\alpha^g, \beta^g), \qquad \alpha, \beta \in \Omega,\ g \in G.$$

Set $\mathrm{Inv}(G) := (\Omega, \mathcal{C})$ where $\mathcal{C} := \mathrm{Orb}(G, \Omega \times \Omega)$. Then

1. $\mathrm{Inv}(G)$ is a coherent configuration (of G),
2. the basic relations of $\mathcal{X}$ are the 2-orbits of G,

Examples. Permutation groups.

Let $G \leq \mathrm{Sym}(\Omega)$ be a permutation group. It acts on $\Omega \times \Omega$:

$$(\alpha, \beta)^g := (\alpha^g, \beta^g), \qquad \alpha, \beta \in \Omega,\ g \in G.$$

Set $\mathrm{Inv}(G) := (\Omega, \mathcal{C})$ where $\mathcal{C} := \mathrm{Orb}(G, \Omega \times \Omega)$. Then

1. $\mathrm{Inv}(G)$ is a coherent configuration (of G),
2. the basic relations of $\mathcal{X}$ are the 2-orbits of G,
3. $\Phi(\mathcal{X}) = \mathrm{Orb}(G, \Omega)$, in particular $\mathcal{X}$ is a scheme iff G is transitive;

Examples. Permutation groups.

Let $G \leq \mathrm{Sym}(\Omega)$ be a permutation group. It acts on $\Omega \times \Omega$:

$$(\alpha, \beta)^g := (\alpha^g, \beta^g), \qquad \alpha, \beta \in \Omega,\ g \in G.$$

Set $\mathrm{Inv}(G) := (\Omega, \mathcal{C})$ where $\mathcal{C} := \mathrm{Orb}(G, \Omega \times \Omega)$. Then

1. $\mathrm{Inv}(G)$ is a coherent configuration (of G),
2. the basic relations of $\mathcal{X}$ are the 2-orbits of G,
3. $\Phi(\mathcal{X}) = \mathrm{Orb}(G, \Omega)$, in particular $\mathcal{X}$ is a scheme iff G is transitive;

Definition.

A coherent configuration $\mathcal{X}$ is called schurian if $\mathcal{X} = \mathrm{Inv}(G)$ for some group G.

Schurity problem

Given a coherent configuration $\mathcal{X}$, find whether it is schurian.

Galois correspondence.

Definition

Let $\mathcal{X} = (\Omega, \mathcal{C}), \mathcal{X}' = (\Omega, \mathcal{C}')$ be two coherent configuratios. We say that $\mathcal{X}$ is a fusion of $\mathcal{X}'$ (equivalently $\mathcal{X}'$ is a fission of $\mathcal{X}$), notation $\mathcal{X} \sqsubseteq \mathcal{X}'$ if $\mathcal{C} \sqsubseteq \mathcal{C}'$.

Galois correspondence.

Definition

Let $\mathcal{X} = (\Omega, \mathcal{C}), \mathcal{X}' = (\Omega, \mathcal{C}')$ be two coherent configuratios. We say that $\mathcal{X}$ is a fusion of $\mathcal{X}'$ (equivalently $\mathcal{X}'$ is a fission of $\mathcal{X}$), notation $\mathcal{X} \sqsubseteq \mathcal{X}'$ if $\mathcal{C} \sqsubseteq \mathcal{C}'$.

Proposition

Let $\mathcal{X}, \mathcal{X}'$ be two coherent configurations defined on Ω and $G, H \leq \mathrm{Sym}(\Omega)$ arbitrary subgroups. Then

- $\mathcal{X} \sqsubseteq \mathcal{X}' \implies \mathrm{Aut}(\mathcal{X}) \geq \mathrm{Aut}(\mathcal{X}')$;
- $H \leq G \implies \mathrm{Inv}(H) \sqsupseteq \mathrm{Inv}(G)$;
- $G \leq \mathrm{Aut}(\mathrm{Inv}(G))$;
- $\mathcal{X} \sqsubseteq \mathrm{Inv}(\mathrm{Aut}(\mathcal{X}))$

Galois closed objects.

Definition

The group $G^{(2)} := \mathsf{Aut}(\mathsf{Inv}(G))$ is called a 2-closure of $G \leq \mathsf{Sym}(\Omega)$. A group is called 2-closed if $G = G^{(2)}$.

Definition

Given a coherent configuration $\mathcal{X} = (\Omega, \mathcal{C})$, the configuration $\mathrm{Sch}(\mathcal{X}) := \mathsf{Inv}(\mathsf{Aut}(\mathcal{X}))$ is called a Schurian closure of $\mathcal{X}$. A configuration $\mathcal{X}$ is schurian iff $\mathrm{Sch}(\mathcal{X}) = \mathcal{X}$.

Theorem

The mappings $(\mathsf{Aut}, \mathsf{Inv})$ are bijections between 2-closed subgroups of $\mathsf{Sym}(\Omega)$ and schurian coherent configurations defined on Ω.

Theorem.

The ISO is polynomially equivalent to the problem of finding the schurian closure of a coherent configuration.

Galois closed objects.

Definition

The group $G^{(2)} := \mathsf{Aut}(\mathsf{Inv}(G))$ is called a 2-closure of $G \leq \mathsf{Sym}(\Omega)$. A group is called 2-closed if $G = G^{(2)}$.

Definition

Given a coherent configuration $\mathcal{X} = (\Omega, \mathcal{C})$, the configuration $\mathrm{Sch}(\mathcal{X}) := \mathsf{Inv}(\mathsf{Aut}(\mathcal{X}))$ is called a Schurian closure of $\mathcal{X}$. A configuration $\mathcal{X}$ is schurian iff $\mathrm{Sch}(\mathcal{X}) = \mathcal{X}$.

Theorem

The mappings $(\mathsf{Aut}, \mathsf{Inv})$ are bijections between 2-closed subgroups of $\mathsf{Sym}(\Omega)$ and schurian coherent configurations defined on Ω.

Theorem.

The ISO is polynomially equivalent to the problem of finding the schurian closure of a coherent configuration.