Group Irregularity Strength of Graphs

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Notation

- G - simple graph with no components of order less than 3
- $E(G)$ - the edge set of G
- $V(G)$ - the vertex set of G
- $n = |V(G)|$
- $\mathcal{G}$ - Abelian group, for convenience: 0, $2a$, $-a$, $a-b\dots$

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$s(G)$: Definition

Assign positive integer $w(e) \leq s$ to every edge $e \in E(G)$.

- For every vertex $v \in V(G)$ the *weighted degree* is defined as:

$$wd(v) = \sum_{e \ni v} w(e).$$

- w is irregular if for $v \neq u$ we have $wd(v) \neq wd(u)$.
- ***Irregularity strength*** $s(G)$**:** the lowest s that allows some irregular labeling.

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Introduced by G. Chartrand, M.S. Jacobson, J. Lehel, O.R. Oellermann, S. Ruiz, F. Saba, 1988.

$s(G)$: Some results

- Lower bound:

$$s(G) \geq \max_{1 \leq i \leq \Delta} \frac{n_i + i - 1}{i}$$

- Best upper bound (M. Kalkowski, M. Karoński, F. Pfender, 2009):

$$s(G) \leq \left\lceil \frac{6n}{\delta} \right\rceil$$

- Exact values for some families of graphs (e.g. cycles, grids, some kinds of trees, circulant graphs).

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Labellings with finite Abelian groups

- Harmonious graphs (Graham and Sloane, Beals et al., Żak).
- A-cordial labellings (Hovey).
- Edge-magic total labellings (Cavenagh et al.).
- Group distance magic graphs (Froncek).
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$s_g(G)$: Main Result

Theorem

Let G be arbitrary connected graph of order $n \geq 3$. Then

$$s_g(G) = \begin{cases} n+2 & \text{when } G \cong K_{1,3^{2q+1}-2} \text{ for some integer } q \geq 1 \\ n+1 & \text{when } n \equiv 2 \pmod 4 \wedge G \ncong K_{1,3^{2q+1}-2} \\ n & \text{otherwise} \end{cases}$$

$s_g(G)$: Lower bound

Lemma

Let G be of order n, if $n \equiv 2 \pmod 4$, then $s_g(G) \geq n+1$.

Proof.

Assume we can use some $\mathcal{G}$ of order $2(2k+1)$. Obviously $\mathcal{G} = Z_2 \times \mathcal{G}_1$. There are $2k+1$ elements $(1, a)$ where $a \in \mathcal{G}_1$ and we have to use all of them. On the other hand

$$\sum_{x \in G} w(x) = (0, b)$$

for some $b \in \mathcal{G}_1$. Contradiction. □

$s_g(K_{1,n-1})$

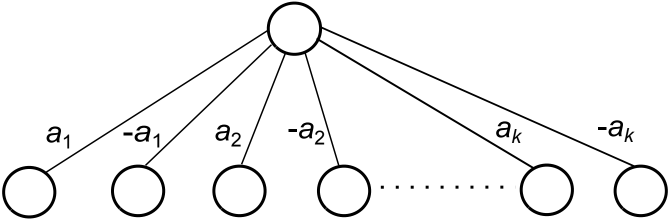
Lemma

Let $K_{1,n-1}$ be a star with $n-1$ pendant vertices. Then

$$s_g(K_{1,n-1}) = \begin{cases} n+2 & \text{when } n \equiv 2 \pmod 4 \wedge n = 3^q - 2 \\ n+1 & \text{when } n \equiv 2 \pmod 4 \wedge n \neq 3^q - 2 \\ n & \text{otherwise} \end{cases}$$

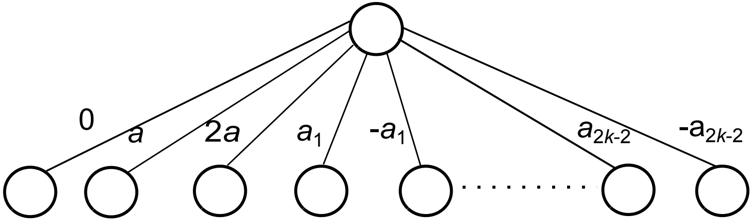
$s_g(K_{1,n-1})$ - proof

Case $n = 2k + 1$:



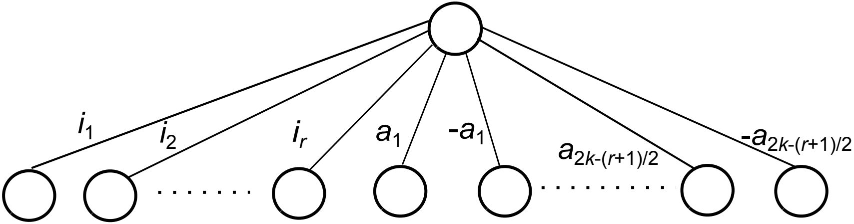
$s_g(K_{1,n-1})$ - proof

Case $n = 4k$, one involution a - there is a subgroup $\{0, a, 2a, 3a\}$:



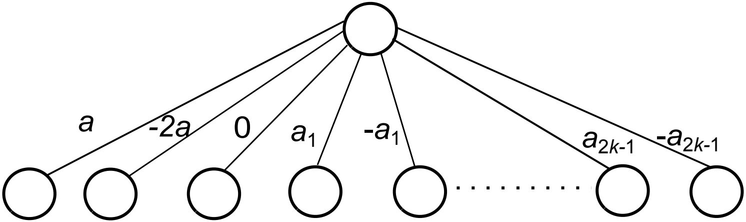
$s_g(K_{1,n-1})$ - proof

Case $n = 4k$, r involutions $i_1, i_2, \ldots, i_r$:



$s_g(K_{1,n-1})$ - proof

Case $n = 4k + 2$, there exists element a of order more than 3:



$s_g(K_{1,n-1})$

Case $n = 4k + 2$, $4k + 3 = 3^q$, all the elements have order 3:

- $\mathcal{G}=Z_3 \times Z_3 \times \cdots \times Z_3$, we do not use exactly two **distinct** elements a and b.
- Sum at the central vertex: $-a - b$, has to be equal either a or b implies $a = b$, contradiction.
- Possible to use $\mathcal{G}$ of order $4k + 4$ as there exists $a \in \mathcal{G}$ of order more than 2 (otherwise $4k + 4 = 2^p$ - contradiction to the Mihăilescu Theorem). We use all but 0, a and $-a$.

$s_g(T)$

Lemma

Let T be arbitrary tree on $n \geq 3$ vertices not being a star. Then

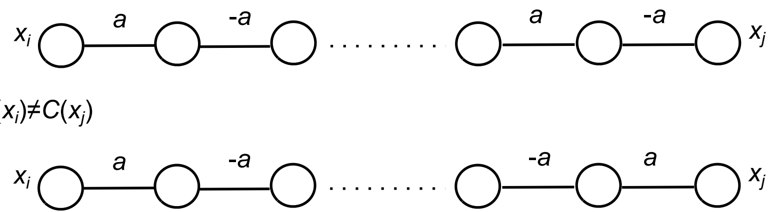
$$s_g(T) = \begin{cases} n+1 & \text{when } n \equiv 2 \pmod 4 \\ n & \text{otherwise} \end{cases}$$

$s_g(T)$ - proof

Main idea: alternating paths.

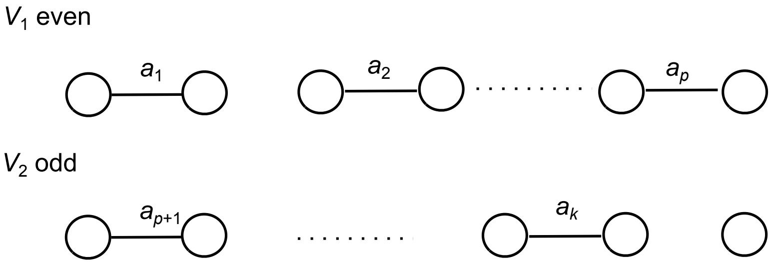
$C(x_i)=C(x_j)$

$C(x_i)\neq C(x_j)$



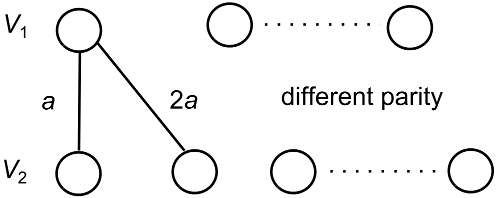
$s_g(T)$ - proof

Case $n = 2k+1$: take $a_1, \ldots, a_k$, $a_i \notin \{a_j, -a_j\}$.



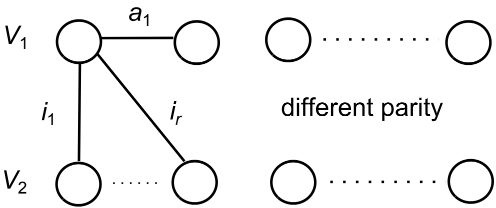
$s_g(T)$ - proof

Case $n = 4k$, one involution - subgroup $\{0, a, 2a, 3a\}$, reduction:



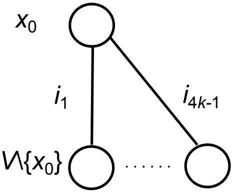
$s_g(T)$ - proof

Case $n = 4k$, $r \leq n/2$ involutions:



$s_g(T)$ - proof

Case $n = 4k$, $r = n - 1$ involutions, $\mathcal{G} = Z_2 \times \cdots \times Z_2$



$s_g(T)$ - proof

- Case $n = 4k + 2$, colour classes even: use $\mathcal{G}$ without 0.
- Colour classes odd: we label $K_{3,5}$.

Open Problem

Problem

Determine group irregularity strength $s_g(G)$ for not-connected graph G with no component of order less than 3.

Open Problem

Problem

Characterize the graphs G such that if $s_g(G) = s$ then G admits a $\mathcal{G}$-labeling for every group $\mathcal{G}$ of order greater than s.

Observation

Let G be arbitrary connected graph on $n \geq 3$ vertices not being a star. Then G admits $\mathcal{G}'$-irregular labelling for any abelian group $\mathcal{G}'$ of order $k > n$, if $k = 2^p(2m+1)$ and $m \in N$ and ($2m \geq n-1$ or $0 \leq p \leq \lfloor \log_2(n+1) \rfloor$).

Open Problem

Problem

Let G be a simple graph with no components of order less than 3. For any Abelian group $\mathcal{G}$, let $\mathcal{G}^* = \mathcal{G} \setminus \{0\}$. Determine non-zero group irregularity strength ($s_g^*(G)$) of G, i.e. the smallest value of s such that taking any Abelian group $\mathcal{G}$ of order s, there exists a function $f : E(G) \to \mathcal{G}^*$ such that the sums of edge labels in every vertex are distinct.

Some History

- Thu: complete graphs.
- Sat: cycles.
- Mon: trees.
- Wed: VICTORY!

Victory

Thank You

THANK YOU :-)

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