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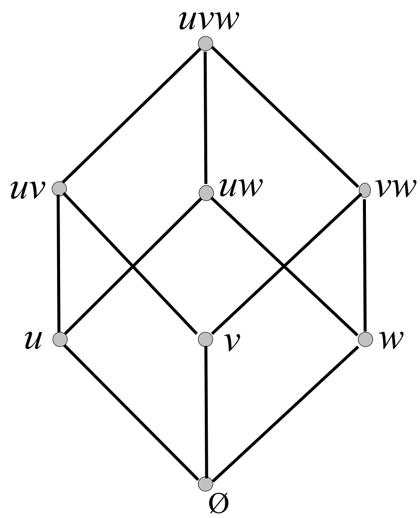
MEDIAN ALGEBRAS and GRAPHS

Koper, 16.5.2011



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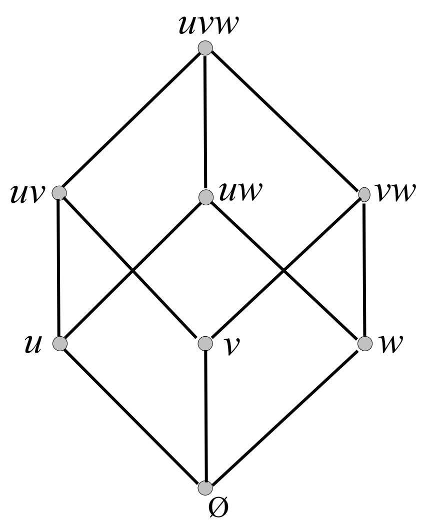
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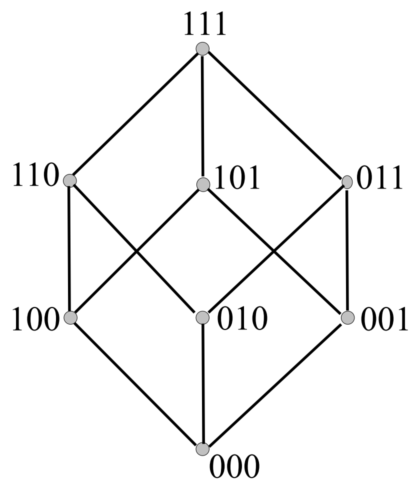
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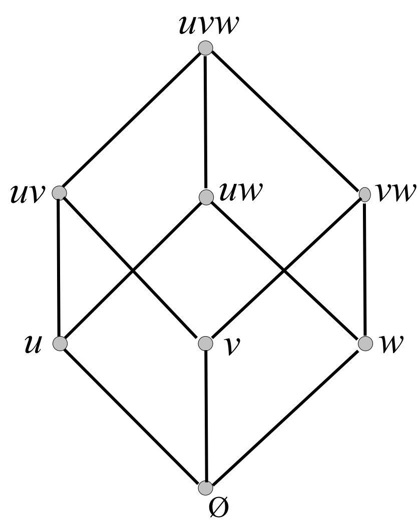

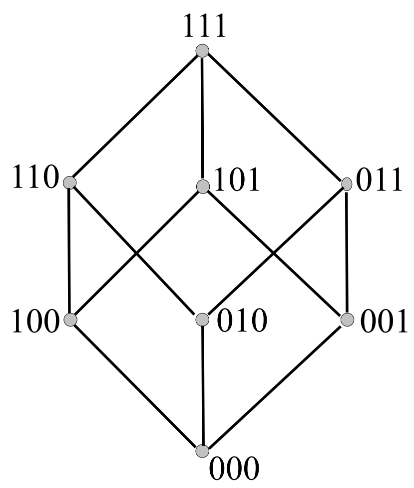

Hypercube or n-*cube* Q_n: $\quad V(Q_n) = \{0,1\}^n$,

$$xy \in E(Q_n) \iff \text{E}!\, i : x_i \neq y_i$$

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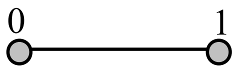
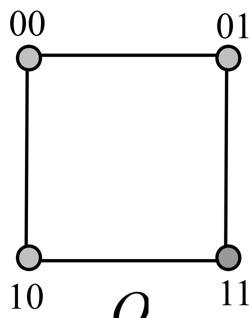
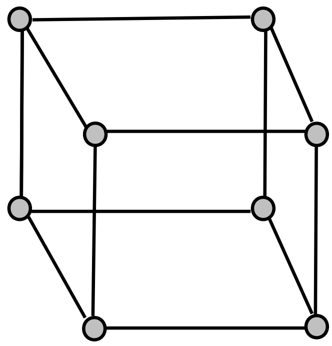


Q_1



Q_2



Q_3

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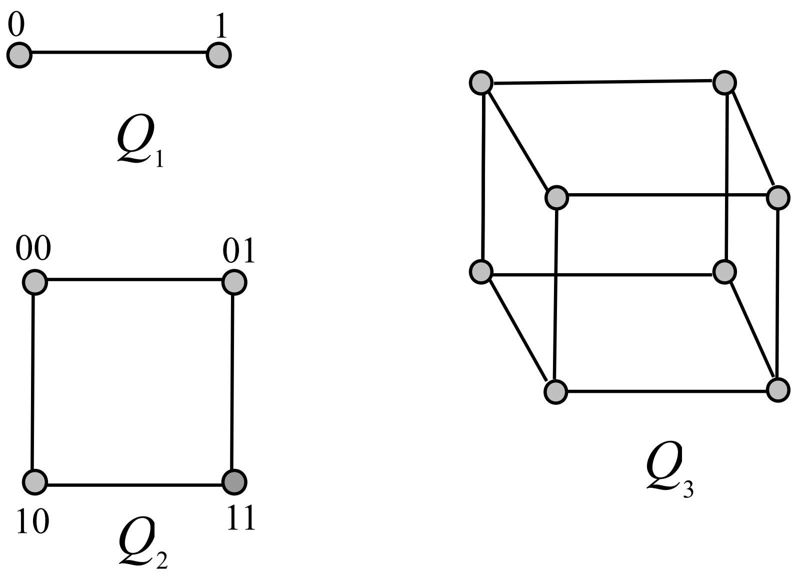
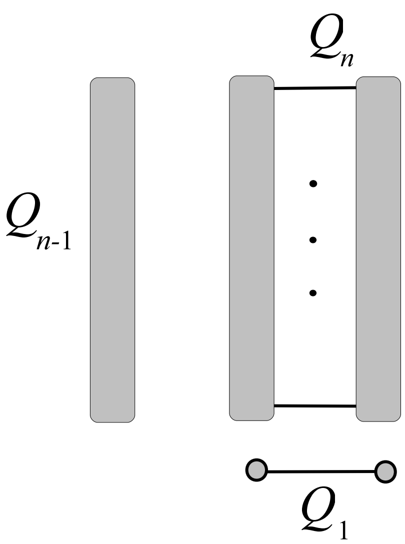


$Q_n = (K_2)^n$



$$Q_n = Q_{n-1} \square Q_1$$

MEDIAN ALGEBRAS

Let (M, m) be a ternary algebraic structure, i.e. M is a set and $m : M \times M \times M \longrightarrow M$ (we use $m(x, y, z) =< xyz >$).

Def. (M, m) is a **median algebra**, if it enjoys the following properties:

(A1) $<xyz>=<\pi(x)\pi(y)\pi(z)>$ commutativity

(A2) $< xxy >= x$ majority strategy

(A3) $< xu < yuz >>=<< xuy > uz >$ associativity

Third axiom alternatively:

(A3') $< xy < uvz >>=<< xyu >< xyv > z >$ distributivity

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BRIEF HISTORY

G. Birkhoff, S. Kiss, A ternary operation in distributive lattices, Bull. Amer. Math. Soc. 53 (1947) 749-752.

M. Sholander, Medians, lattices and trees, Proc. Amer. Math. Soc. 5 (1954) 808-812.

S. Avann, Metric ternary distributive semi-lattices, Proc. Amer. Math. Soc. 12 (1961) 407-414.

J. R. Isbell, Median algebra, Trans. Amer. Math. Soc. 260 (1980) 319-362.

H. M. Mulder, *The Interval Function of a Graph*, Math. Centre Tracts 132, Amsterdam, 1980.

INTERVALS IN MEDIAN ALGEBRAS

Def. **Ideal** of a median algebra (M, m) is a subset $J \subseteq M$ such that for any $x, y \in J$ and any $a \in M$:

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Proposition 1 Every interval in a median algebra is an ideal.

Proposition 2 $I(x, y) = \{u \,|\, <xyu>= u\}$.

Proposition 3 For arbitrary $x, y, z \in M$ we have

$$I(x, y) \cap I(x, z) \cap I(y, z) =\{<xyz>\}.$$

Corollary For arbitrary $x, y, z \in M$ we have

$$|I(x, y) \cap I(x, z) \cap I(y, z)| = 1.$$

MEDIAN GRAPHS

Let (M, m) be a median algebra. Then its *underlying graph* G_M has M as its vertex set, and $x, y \in M$ are adjacent if $I(x, y) = \{x, y\}$.

Proposition 4 For an arbitrary discrete median algebra M its underlying graph G_M is connected and bipartite. An interval $I(x, y)$ in a median algebra coincides with an interval in a graph, induced by the shortest paths metrics.

$$\begin{aligned} I(x, y) &= \{u \mid u \text{ lies on a shortest path between } x \text{ and } y\} \\ &= \{u \mid d(x, y) = d(x, u) + d(u, y)\}. \end{aligned}$$

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Corollary. The underlying graph G_M of a discrete median algebra M is a median graph. The ternary algebraic structure (V, m) in a median graph $G = (V, E)$, where m is defined by

$$m(x, y, z) = I(x, y) \cap I(x, z) \cap I(y, z)$$

is a median algebra.

CONVEX SETS AND IDEALS

Def. A set C is g-**convex**, if for any pair $x, y \in C$ we have $I(x, y) \subseteq C$.

Corollary Convex sets of a median graph are exactly ideals of the corresponding median algebra.

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Def. Let G be a (median) graph and ab an edge in G. A set

$$W_{ab} = \{x \mid d(x, a) < d(x, b)\}$$

is called **a halfspace**.

Izrek. Every halfspace in a median graph is convex. Moreover, any convex subset in a median graph can be realized as an intersection of halfspaces.

A FEW CHARACTERIZATIONS

A connected graph is a median graph if and only if it ...

(Mulder, 1978) ... is a median-closed isometric subgraph of a hypercube;

(Isbell, 1980) ... it can be obtained by a sequence of convex amalgamations from hypercubes;

(Bandelt, 1984) ... is a retract of a hypercube;

(Chung, Graham, Saks, 1987) ... has an optimal solution for dynamic location problem;

(Tardif, 1996) ... its intervals enjoy Helly property;

(Chepoi, 2000) ... is the undelying graph (1-skeleton) of some CAT(0) polyhedral cube complex.

(B., 2003): ... G is bipartite and every halfspace W_{ab} in G is gated.

MEDIAN GRAPHS in COMBINATORIAL and GEOMETRIC GROUP THEORY

M. Gromov, *Hyperbolic groups*, in: S. Gersten (Ed.), Essays in Group Theory, in: Math. Sci. Res. Inst. Publ., vol. 8, Springer-Verlag, Berlin, 1987, pp. 75263.

Thm. (Gromov, 1987). Polyhedral cubical complex $|C|$ with intrinsic l_2-metrics is CAT(0) if and only if $|C|$ is simply connected and enjoys the condition: if three $(k+2)$-cubes from $|C|$ share a common k-cube and pair-wise share $(k+1)$-cubes, they are included in a $(k+3)$-cube of a complex $|C|$.

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G. A. Niblo, L. D. Reeves, Groups acting on CAT(0) cube complexes, Geom. & Topology 1 (1997), 7 pp.

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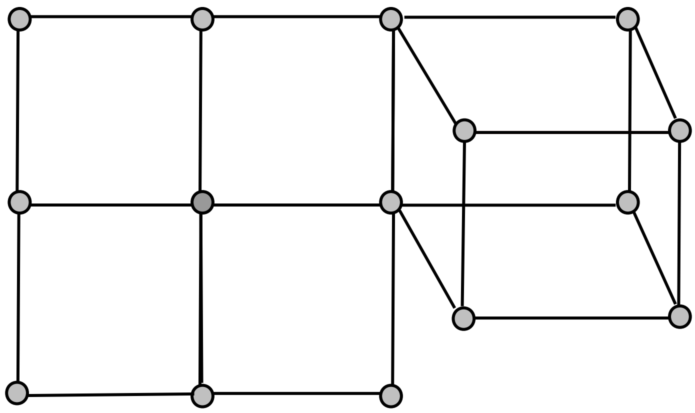
Thm. (Isbell, 1980) G is a median graph if and only if it can be obtained by a sequence of convex amalgamations from hypercubes.



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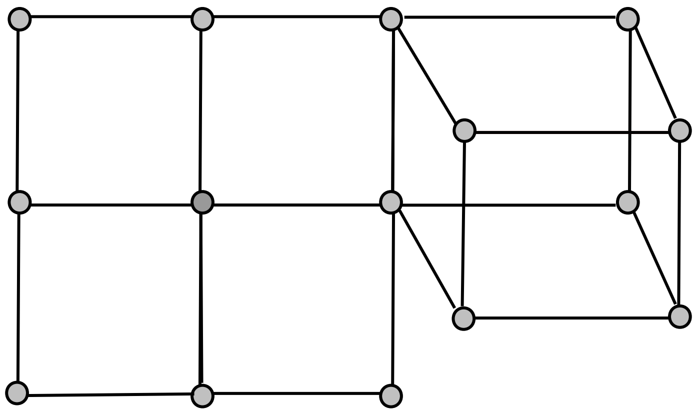

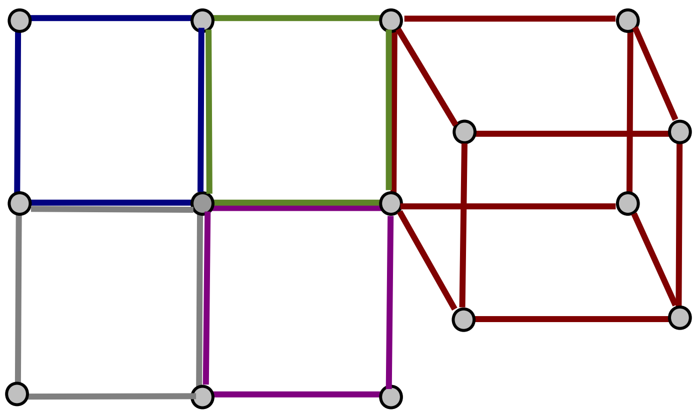

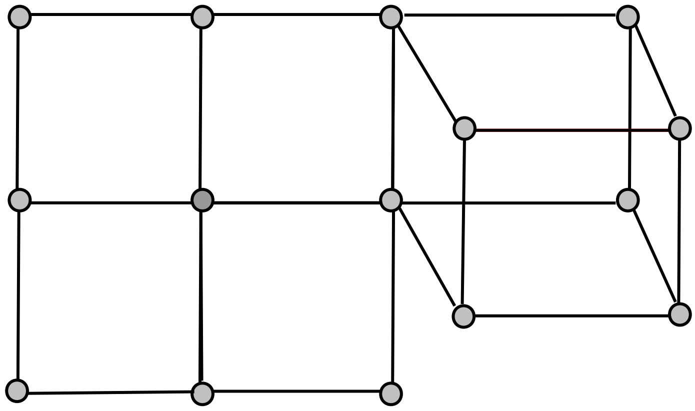

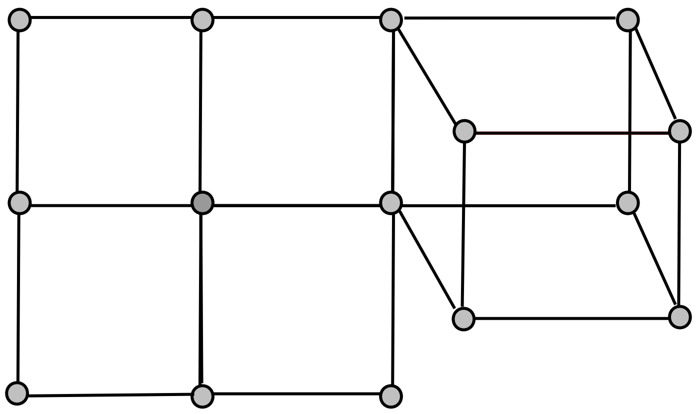

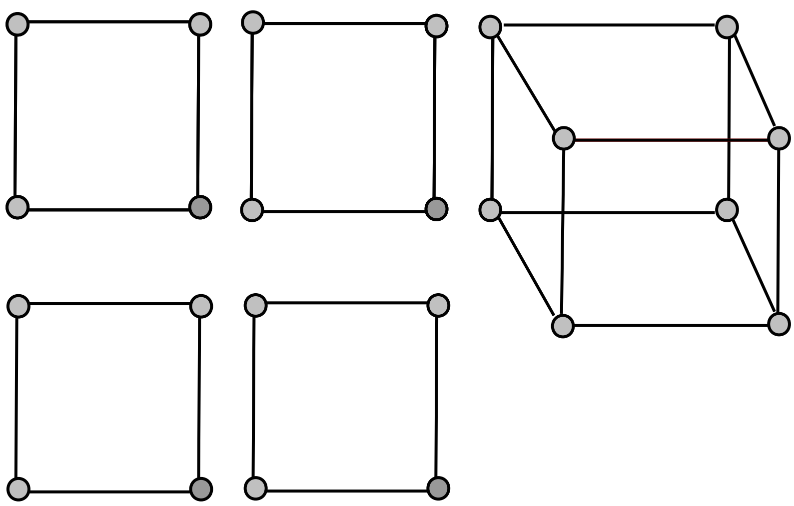

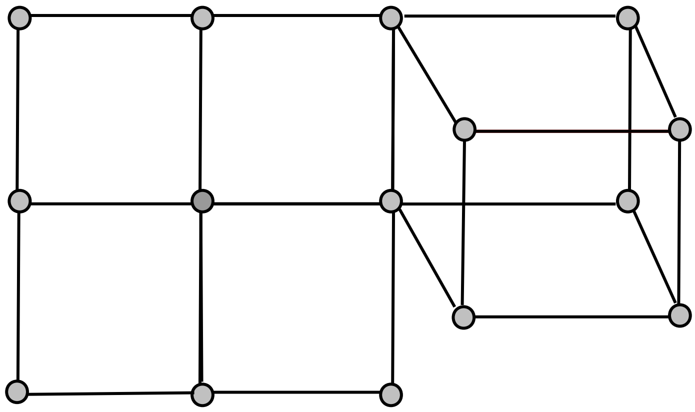

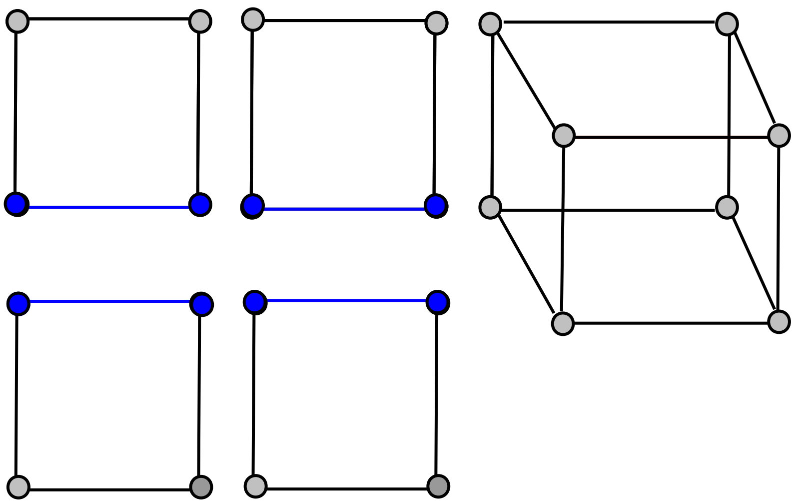

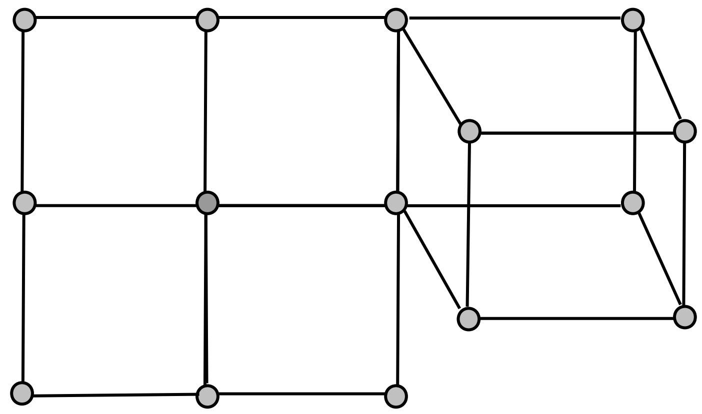

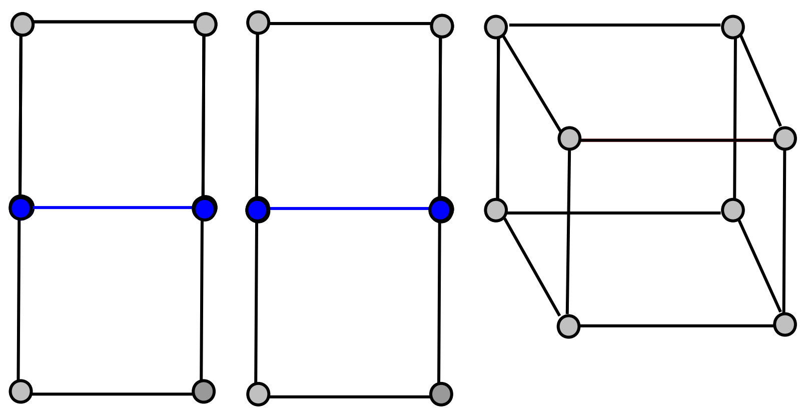

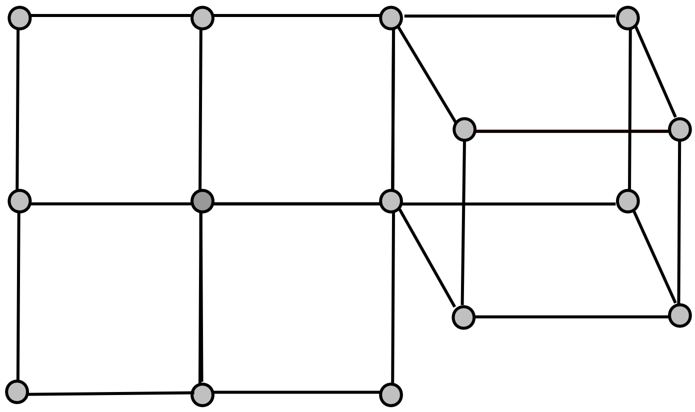

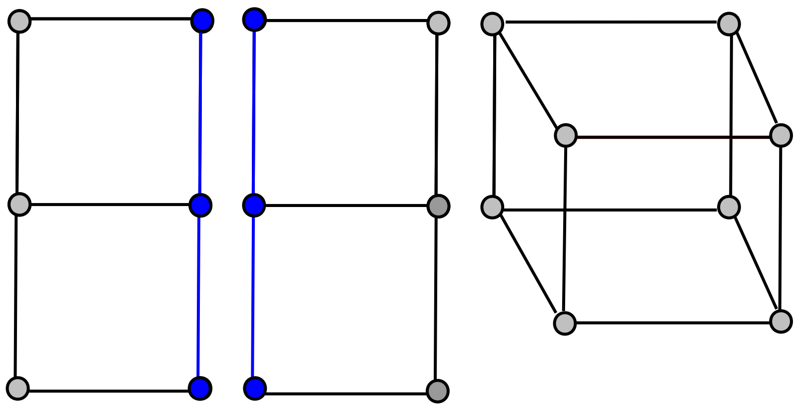

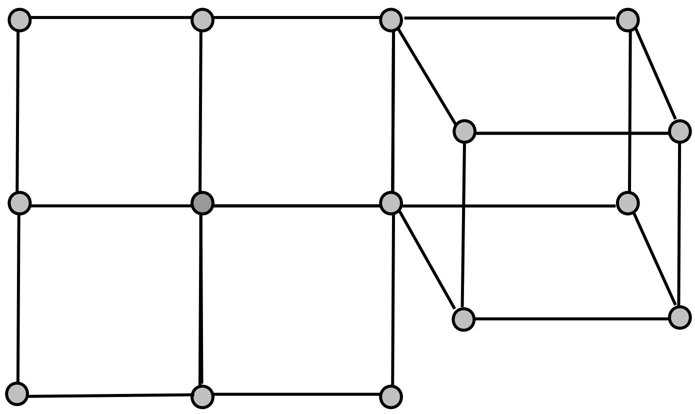

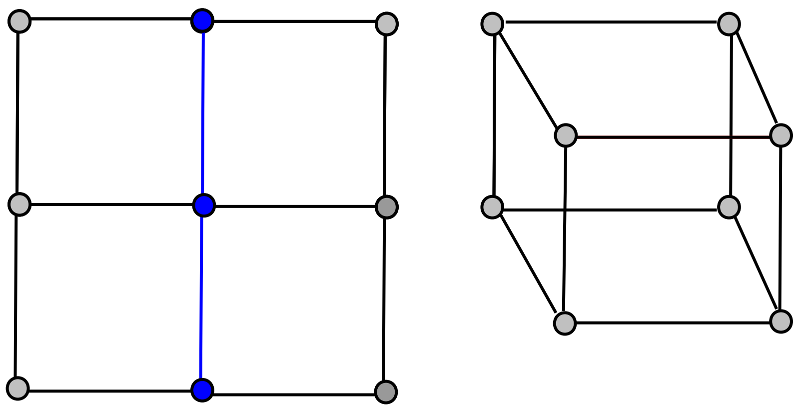

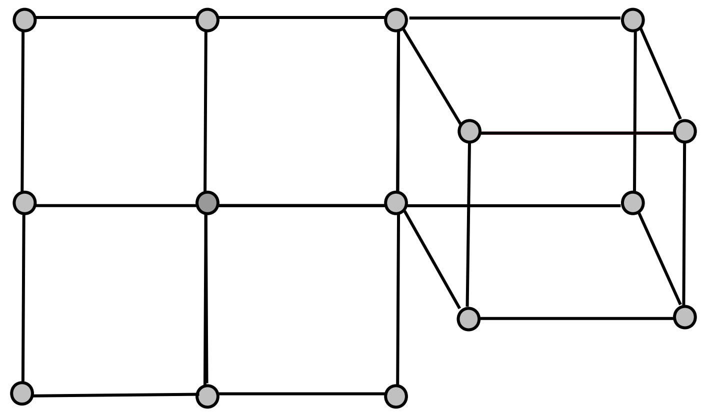

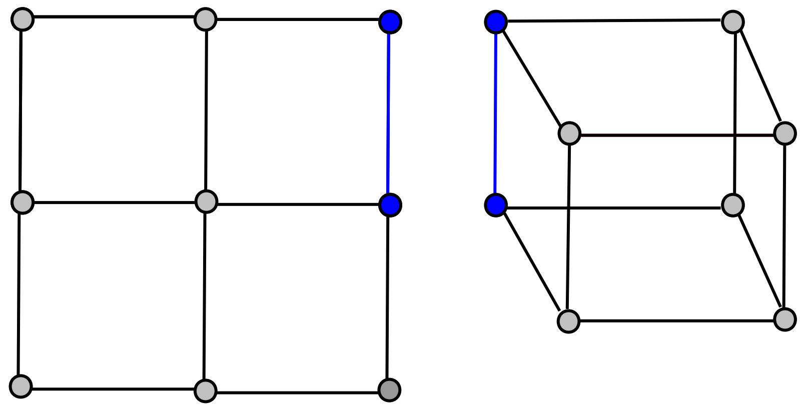

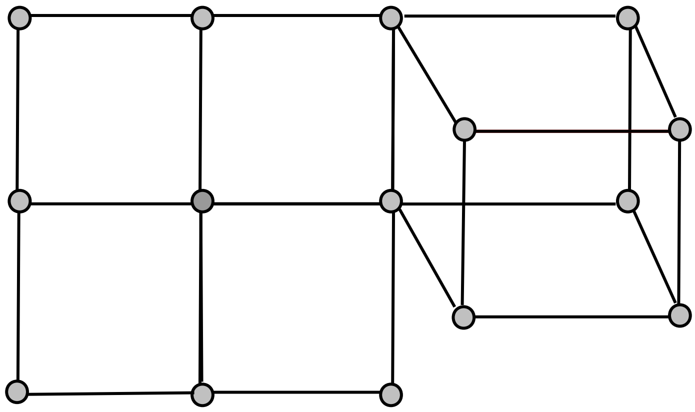

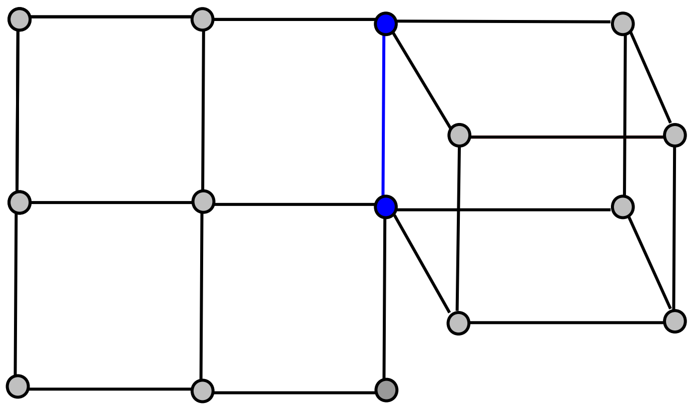

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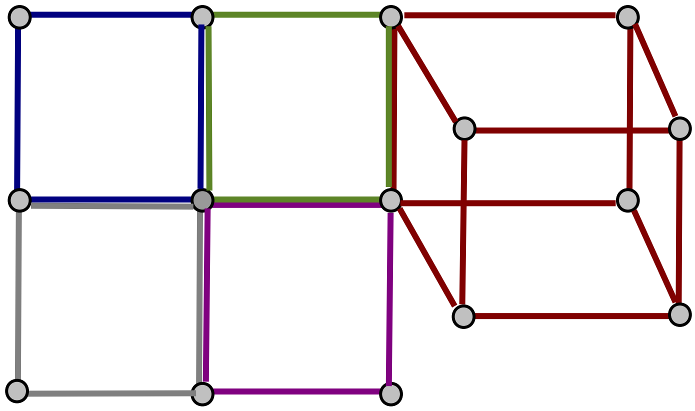

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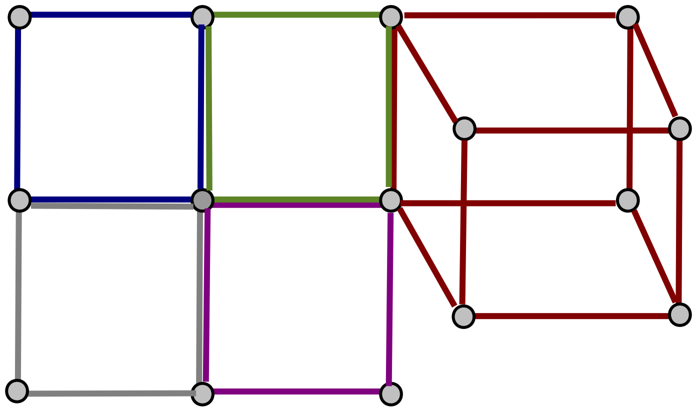


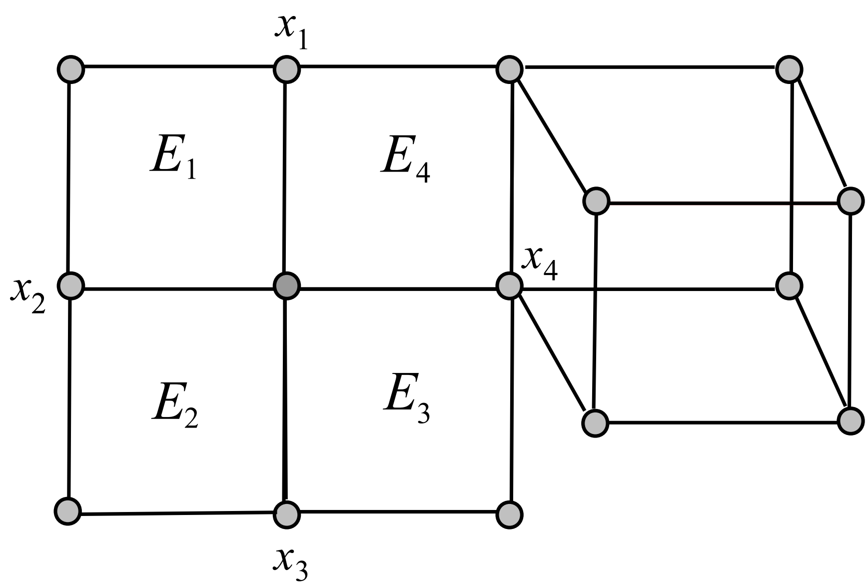
Open problem: Is every Helly graph H the intersection graph of maximal hypercubes of some median graph?

(Is there a median graph G, such that $q(G) = H$?)

ACYCLIC CUBICAL COMPLEXES

Def. An (abstract) *cubical complex* $\mathcal{K}$ is a set of (graphic) cubes closed for subcubes and nonempty intersections. In the *underlying graph* of a cubical complex $\mathcal{K}$ two vertices of $\mathcal{K}$ are adjacent whenever they constitute a 1-dimensional cube.

Cycle of a complex $\mathcal{K}$ is $x_1, E_1, x_2, E_2, \ldots, x_k, E_k, x_1$ where x_i are distinct vertices and E_i distinct cubes of $\mathcal{K}$, such that $x_i, x_{i+1} \in E_i$ for $i = 1, \ldots, k$ (mod k), and no member of $\mathcal{K}$ includes three distinct vertices of a cycle. Cubical complex $\mathcal{K}$ is *conformal* if any set of vertices that are pair-wisely in a common cube, is included in a cube of $\mathcal{K}$. Complex $\mathcal{K}$ is *acyclic* if it is conformal and has no cycles.



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- $\mathcal{K}$ is acyclic,
- simplicial complex $\mathcal{K}^{\Delta}$ is acyclic,
- $\mathcal{K}$ enjoys a peripheral cube contraction scheme,
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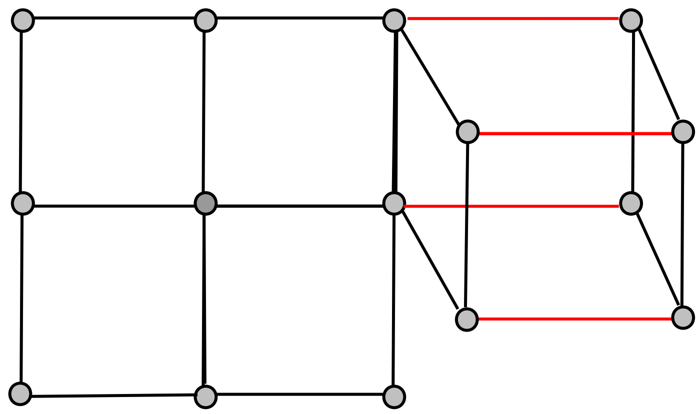

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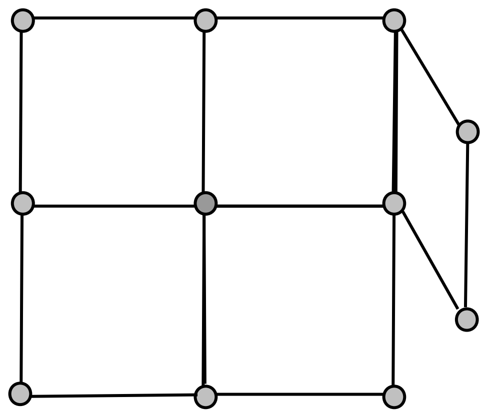

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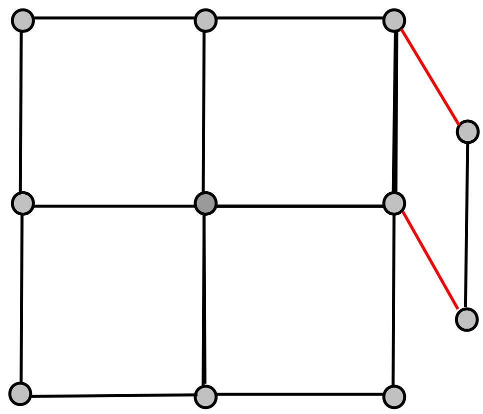

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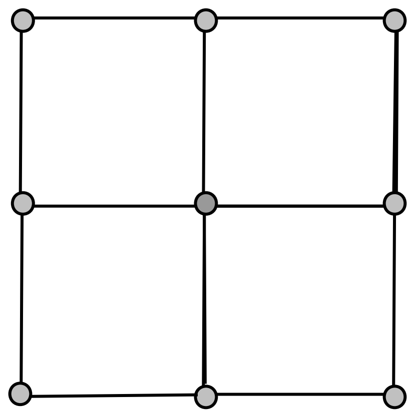

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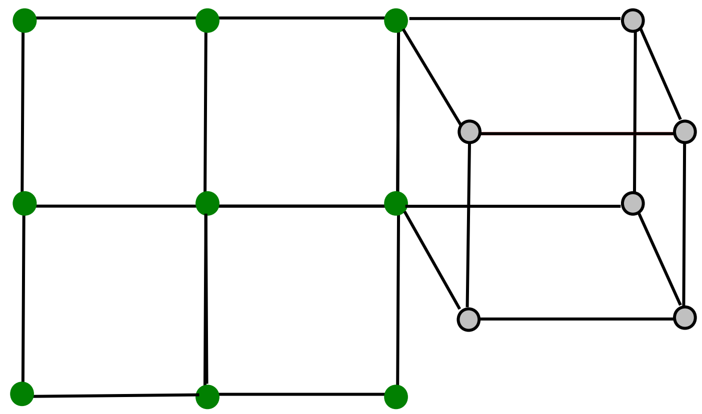

Corollary. Let G be the graph of an acyclic cubical complex. The intersection graph of maximal hypercubes $q(G)$ of G is a dually chordal graph (i.e. clique graph of a chordal graph; or, the underlying graph of a hypertree (arboreal hypergraph)).

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Proof

Idea 1: Elimination scheme of dually chordal graphs.

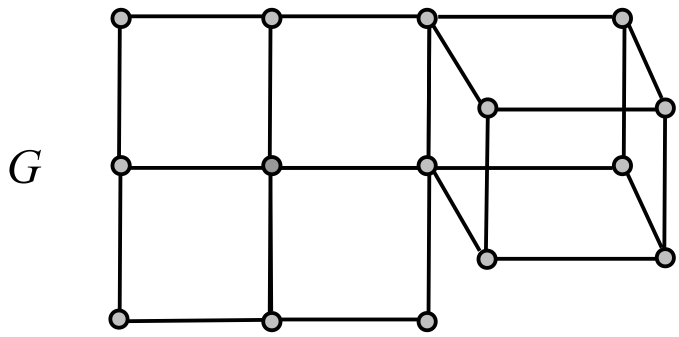
Idea 2: Maximal-2-intersection cube graph of an acyclic cubical complex is a block graph.

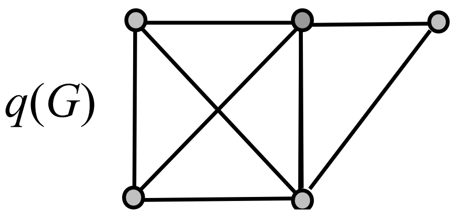
BACK TO MEDIAN GRAPHS

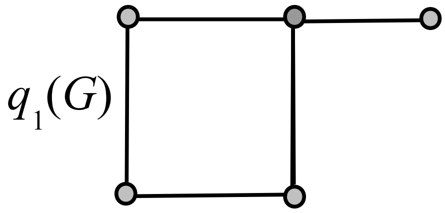
B. Brešar, T. Kraner Šumenjak, Cube intersection concepts in median graphs, Discrete Math. 309 (2009) 2990-2997.

Def. Let G be a median graph and $k \geq 0$. The graph $q_k(G)$ has maximal hypercubes of G as its vertices and $H_x, H_y \in V(q_k(G))$ are adjacent if and only if $H_x \cap H_y$ contains a k-cube.

Note: $q_0(G) = q(G)$.







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Part 1. Idea: conformality of 1-cubes in hypercubes.

Lemma: Let S be a set of edges in a median graph G. If edges from S pair-wise belong to a common hypercube then there exists a hypercube that contains all edges from S.

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Part 2. Idea: simplex graph $\kappa(G)$ of G.

$$q_1(\kappa(G)) = K(G)$$

REALIZATION THEOREM - more generally

Thm. For any median graph G, $q_1(G)$ is a clique-graph (graph realizable as the clique graph of some graph).

For any clique-graph H there exists a median graph G such that $q_1(G) = H$.

Thm. Let $k \geq 1$. For any median graph G, $q_k(G)$ is a clique-graph. For any clique-graph H there exists a median graph G such that $q_k(G) = H$.

MAXIMAL 2-INTERSECTION CUBE GRAPHS

Def. Let G be a median graph. The *maximal* 2*-intersection cube graph* $\mathcal{Q}_{\mathrm{m2}}(G)$ of G has maximal hypercubes of G as its vertices, and two vertices are adjacent if the corresponding hypercubes have a maximal 2-intersection in common.

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For any diamond-free graph H there exists a median graph G such that $\mathcal{Q}_{\mathrm{m2}}(G) = H$.

REALIZATION THEOREMS and A CONJECTURE

Thm. Let $k \geq 1$. For any median graph G, $q_k(G)$ is a clique-graph.

For any clique-graph H there exists a median graph G such that $q_k(G) = H$.

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For any diamond-free graph H there exists a median graph G such that $\mathcal{Q}_{\mathrm{m2}}(G) = H$.

Thm. For any median graph G the graph $q(G)$ is a Helly graph.

Conjecture. For any Helly graph H there is a median graph G such that $q(G) = H$.