Width parameters and graph classes: the case of mim-width

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Joint work:

Bounding the Mim-Width of Hereditary Graph Classes

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Abstract

A large number of NP-hard graph problems become polynomial-time solvable on graph classes where the mim-width is bounded and quickly computable. Hence, when solving such problems on special graph classes, it is helpful to know whether the graph class under consideration has these two properties. We first extend the toolkit for proving (un)boundedness of mim-width of graph classes. This enables us to initiate a systematic study into bounding mim-width from the perspective of hereditary graph classes.

We show that for a given graph H, the class of H-free graphs has bounded mim-width if and only if it has bounded clique-width. We then show that the same is not true for (H_1, H_2)-free graphs. To be more precise, for graphs G_1 and G_2 each having r vertices, let $G_1 \boxminus G_2$ be the graph obtained from the disjoint union of G_1 and G_2 by adding r edges between G_1 and G_2 that form a perfect matching. Let (H_1, H_2) be one of the following:

- $(K_r \boxminus rP_1, 2P_2)$, for $r \geq 3$
- (K_r, tP_2), for $r \geq 3$, $t \geq 2$ and $r + t \geq 6$
- $(K_r \boxminus K_r, sP_1 + P_2)$, for $r \geq 2$, $s \geq 2$ and $r + s \geq 5$
- $(K_{1,3}, 2P_2)$ or $(\overline{C_4 + P_1}, 2P_1 + P_2)$.

It is known that each of the above classes of (H_1, H_2)-free graphs has unbounded clique-width. We prove that each of them has bounded mim-width. As a consequence, we classified all pairs (H_1, H_2) with $|V(H_1)| + |V(H_2)| \leq 8$. Moreover, we prove that the mim-width for each of the above classes of (H_1, H_2)-free graphs is *quickly computable*. That is, we prove there is a polynomial-time algorithm for computing a branch decomposition of constant mim-width. For the first three (infinite) families of classes of (H_1, H_2)-free graphs, we show that the mim-width of any branch decomposition is bounded by a constant, and hence it suffices to compute one arbitrarily. For the latter two cases, we show how to compute a branch decomposition of constant mim-width in polynomial time. Hence, these results have algorithmic implications: when the input is restricted to such a class of (H_1, H_2)-free graphs, many problems become polynomial-time solvable, including classical problems such as k-Colouring and Independent Set, domination-type problems known as LC-VSVP problems, and distance versions of LC-VSVP problems, to name just a few. We also prove a number of new results showing that, for certain H_1 and H_2, the class of (H_1, H_2)-free graphs has unbounded mim-width.

Boundedness of clique-width implies boundedness of mim-width. By combining our new results for mim-width with the known bounded cases for clique-width, we present summary theorems of the current state of the art for the boundedness of mim-width for (H_1, H_2)-free graphs. In particular, we have classified all pairs (H_1, H_2) where H_1 and H_2 are connected graphs, except for one remaining infinite family and a few isolated cases.

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Semitotal Domination: New hardness results and a polynomial-time algorithm for graphs of bounded mim-width

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ABSTRACT

A semitotal dominating set of a graph G with no isolated vertex is a dominating set D of G such that every vertex in D is within distance two of another vertex in D. The minimum size $\gamma_{t2}(G)$ of a semitotal dominating set of G is squeezed between the domination number $\gamma(G)$ and the total domination number $\gamma_t(G)$.

Semitotal Dominating Set is the problem of finding, given a graph G, a semitotal dominating set of G of size $\gamma_{t2}(G)$. In this paper, we continue the systematic study on the computational complexity of this problem when restricted to special graph classes. In particular, we show that it is solvable in polynomial time for the class of graphs of bounded mim-width by a reduction to Total Dominating Set and we provide several approximation lower bounds for subclasses of subcubic graphs. Moreover, we obtain complexity dichotomies in monogenic classes for the decision versions of Semitotal Dominating Set and Total Dominating Set.

Finally, we show that it is NP-complete to recognise the graphs such that $\gamma_{t2}(G) = \gamma_t(G)$ and those such that $\gamma(G) = \gamma_{t2}(G)$, even if restricted to be planar and with maximum degree at most 4, and we provide forbidden induced subgraph characterisations for the graphs hereditarily satisfying either of these two equalities.

Motivation

The following NP-hard problems are polynomial-time solvable on **triad-convex** graphs:

- DOMINATING SET (Pandey and Panda 2019)
- INDEPENDENT DOMINATING SET (Lu et al. 2013)
- CONNECTED DOMINATING SET (Liu et al. 2015)
- DOMINATING INDUCED MATCHING (Panda and Chaudhary 2019)
- FEEDBACK VERTEX SET (Jiang et al. 2013)

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- Feedback Vertex Set (Jiang et al. 2013)

Theorem

The problems above are polynomial-time solvable on graphs of bounded ***mim-width****.*

Theorem (Brettell, M., Paulusma 2020+)

Triad-convex graphs have bounded ***mim-width****.*

Outline

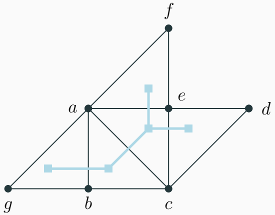
Dynamic Programming and Width Parameters

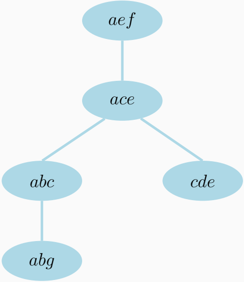
Mim-width

Semitotal Domination

Structural Properties

Treewidth





Tree decomposition of G: pair $(T, \{B_t : t \in V(T)\})$, where T is a tree and $B_t \subseteq V(G)$ for each $t \in V(T)$ (**bag**) satisfying the following:

- each vertex of G is in at least one bag B_t,
- for each $uv \in E(G)$, there exists a bag containing both u and v,
- for each $v \in V(G)$, bags containing v form a subtree of T.

Width of a tree decomposition: $\max_{t \in V(T)} |B_t| - 1$.

Treewidth of G: minimum width of a tree decomposition of G.

DP on tree decompositions: Independent Set

NP-hard to determine the treewidth of a graph (Arnborg et al. 1987).

$2^{O(w^3)} \cdot n$ time algorithm that finds a tree decomposition of width w (Bodlaender 1996).

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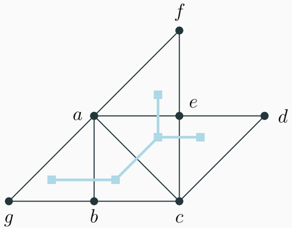
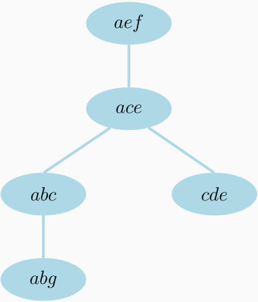
$2^{O(w^3)} \cdot n$ time algorithm that finds a tree decomposition of width w (Bodlaender 1996).





compute $2^{|B_t|} \leq 2^{w+1}$ values for each bag B_t.

$M[t, S]$: max size of independent set $I \subseteq V_t$ (vertices in subtree rooted at t) with $I \cap B_t = S$.

Solve bottom-up: at most $2^{w+1} \cdot n$ subproblems $M[t, S]$, each of them can be solved in constant time (assuming the children are already solved).

Clique-width $\rightsquigarrow$ ***k*-expression**.

The modelling power of clique-width is stronger than that of treewidth:
$\mathrm{cw}(G) \leq 3 \cdot 2^{\mathrm{tw}(G)-1}$ (Corneil and Rotics 2005)

Theorem (Courcelle, Makowsky, Rotics 2000)

Every problem expressible in $MSOL_1$ can be solved in polynomial time on graphs of clique-width at most k, provided a k-expression is given as part of the input.

Theorem (Oum and Seymour 2006)

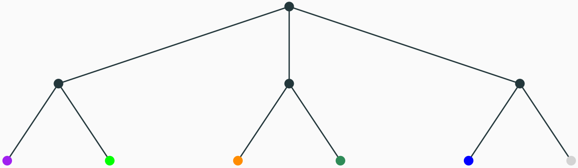
There exists a polynomial-time algorithm for computing a $(2^{3k+2} - 1)$-expression of a graph having clique-width at most k.

Treewidth is equivalent to branch-width: $\mathrm{bw}(G) \leq \mathrm{tw}(G) \leq \frac{2}{3}\mathrm{bw}(G) - 1$ (Robertson and Seymour 1991)

Clique-width is equivalent to rank-width: $\mathrm{rw}(G) \leq \mathrm{cw}(G) \leq 2^{\mathrm{rw}(G)+1} - 1$ (Oum and Seymour 2006)

Decomposition of graphs

- Natural approach to dynamic programming: **recursively partition the vertices** of the graph into two parts.
- **Decomposition** of G can be stored **as a binary tree** whose leaves are in bijection with vertices of G.
- Need to store multiple sub-solutions at each intermediate node $\rightsquigarrow$ **structure of the cuts** is crucial to runtime.



Outline

Branch decomposition for G: (T, δ) where T is subcubic tree and δ is bijection between vertices of G and leaves of T. Each $e \in E(T)$ represents partition $(A_e, \overline{A_e})$ of $V(G)$.

$\mathrm{mimw}_G(T, \delta)$: $\max_{e \in E(T)}$ size of maximum induced matching in $G[A_e, \overline{A_e}]$.

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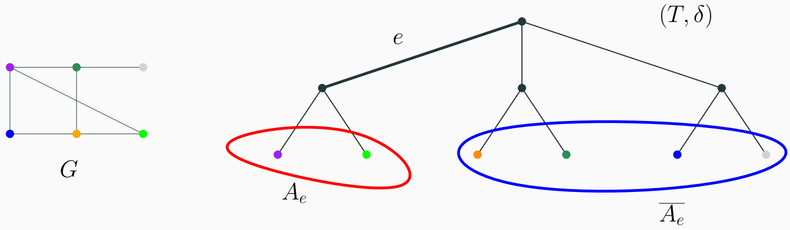

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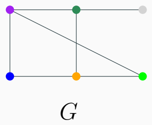

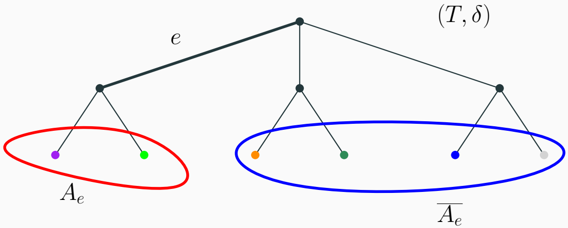

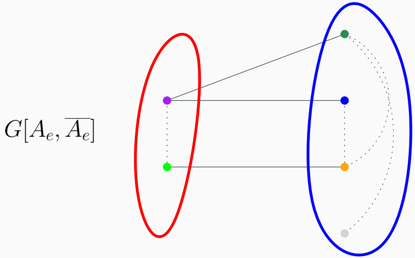

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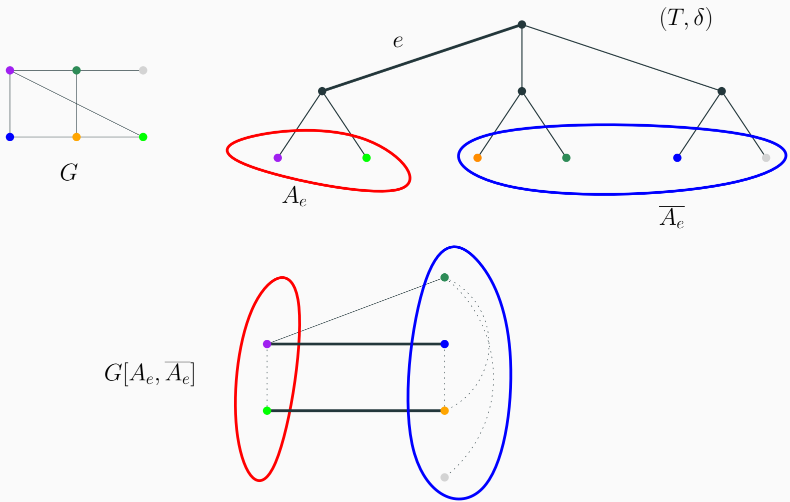

Mim-width of interval graphs

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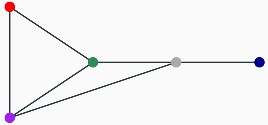

Theorem (Belmonte, Vatshelle 2013)

$\mathrm{mimw}(G) \leq 1$, *for any interval graph* G*. Moreover, a branch decomposition of mim-width at most* 1 *can be computed in linear time.*

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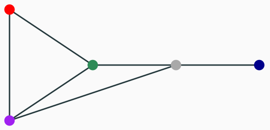

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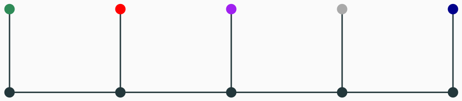

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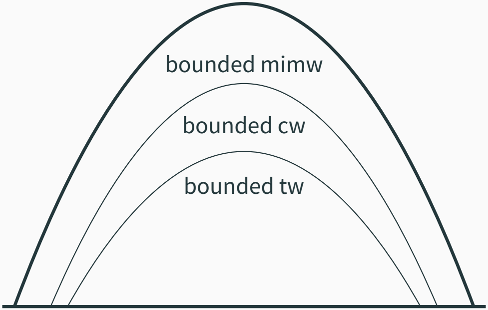

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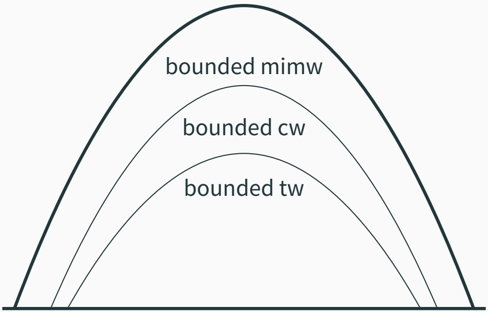

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Locally checkable vertex subset problems (Independent Set, Dominating Set, Total Dominating Set, ...) are in P for classes with bounded mim-width, provided a branch decomposition is given.

(Bui-Xuan et al. 2013)

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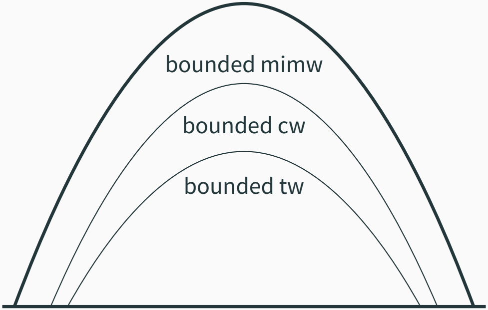

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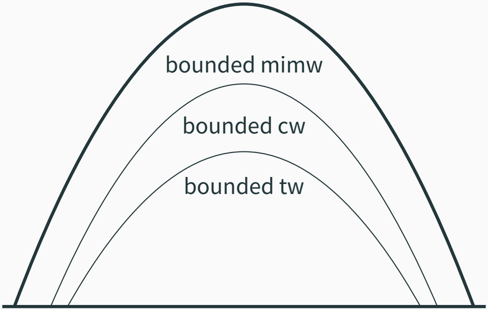

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- Bad News. Computing a branch decomposition with optimal mim-width is NP-complete. Determining the optimal mim-width is unlikely to be in APX.
  (Sæther and Vatshelle 2016)
- Good News. Can find branch decomposition of constant mim-width in polynomial time for: interval graphs, permutation graphs, convex graphs, trapezoid graphs, circular permutation graphs, circular arc graphs, leaf powers, ...
  (Belmonte and Vatshelle 2013)

LCVS problems

Given finite or co-finite subsets σ, ρ of $\mathbb{N}$ and a graph G, $S \subseteq V(G)$ is a **(σ, ρ)-set** if:

- $|N(v) \cap S| \in \sigma$, for each $v \in S$;
- $|N(v) \cap S| \in \rho$, for each $v \in V(G) \setminus S$.

Locally checkable vertex subset problem: find a min or max (σ, ρ)-set in input graph G (Telle and Proskurowski 1997).

Distance-r locally checkable vertex subset problem: replace $N(v)$ with $N^r(v)$ (Jaffke et al. 2020).

σ	ρ	d	Standard name
$\{0\}$	$\mathbb{N}$	1	Independent set $*$
$\mathbb{N}$	$\mathbb{N}^+$	1	Dominating set $**$
$\{0\}$	$\mathbb{N}^+$	1	Maximal Independent set $**$
$\mathbb{N}^+$	$\mathbb{N}^+$	1	Total Dominating set $\star\star$
$\{0\}$	$\{0,1\}$	2	Strong Stable set or 2-Packing
$\{0\}$	$\{1\}$	2	Perfect Code or Efficient Dom. set
$\{0,1\}$	$\{0,1\}$	2	Total Nearly Perfect set
$\{0,1\}$	$\{1\}$	2	Weakly Perfect Dominating set
$\{1\}$	$\{1\}$	2	Total Perfect Dominating set
$\{1\}$	$\mathbb{N}$	2	Induced Matching $\star$
$\{1\}$	$\mathbb{N}^+$	2	Dominating Induced Matching $\star$, $\star\star$
$\mathbb{N}$	$\{1\}$	2	Perfect Dominating set
$\mathbb{N}$	$\{d, d+1, \dots\}$	d	d-Dominating set $\star\star$
$\{d\}$	$\mathbb{N}$	$d+1$	Induced d-Regular Subgraph $\star$
$\{d, d+1, \dots\}$	$\mathbb{N}$	d	Subgraph of Min Degree $\geq d$
$\{0, 1, \dots, d\}$	$\mathbb{N}$	$d+1$	Induced Subg. of Max Degree $\leq d$ $\star$

(Jaffke et al. 2020)

Theorem (Bui-Xuan et al. 2013)

*There is an algorithm that, given a graph G and a branch decomposition (T, δ) for G with $w = \mathrm{mimw}_G(T, \delta)$, solves each **LCVS** problem in $O(n^{4+3dw})$ **time**.*

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*There is an algorithm that for all $r \in \mathbb{N}$, given a graph G and a branch decomposition (T, δ) for G with $w = \mathrm{mimw}_G(T, \delta)$, solves each **distance-r LCVS** problem in $O(n^{4+6dw})$ **time**.*

Solving distance-r LCVS on G is the same as solving distance-1 LCVS on G^r.
Moreover, $\mathrm{mimw}(G^r) \leq 2\mathrm{mimw}(G)$.

LCVS problems and mim-width

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Theorem (Fomin et al. 2018)

Independent Set *and* Dominating Set *are* W[1]***-hard parameterized by*** $\mathrm{mimw}(G)$ ***and solution size***.

Beyond LCVS problems

- Longest Induced Path (Jaffke et al. 2020)
- Induced Disjoint Paths (Jaffke et al. 2020)
- Feedback Vertex Set (Jaffke et al. 2020)
- Subset Feedback Vertex Set (Bergougnoux et al. 2020)
- Node Multiway Cut (Bergougnoux et al. 2020)
- Connected variants of LCVS problems (Bergougnoux and Kanté 2019)
- Semitotal Dominating Set (Galby, M., Ries 2020)

Outline

Domination parameters

- **dominating set**: $S \subseteq V(G)$ such that each vertex in $V(G) \setminus S$ has a neighbour in S.

For a graph with no isolated vertex:

- **total dominating set**: a dominating set S such that each vertex in S has a neighbour in S.
- **semitotal dominating set**: a dominating set S such that each vertex in S is within distance two of another.

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$$\gamma(G) \leq \gamma_{t2}(G) \leq \gamma_t(G)$$

From semitotal to total

Theorem (Galby, M., Ries 2020)

There is an algorithm that, given a graph G and a branch decomposition (T, δ) for G with $w = \mathrm{mimw}_G(T, \delta)$, solves Semitotal Dominating Set *in $O(n^{4+6w})$ time.*

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Idea: Find a bounded mim-width preserving reduction from Semitotal Dominating Set to Total Dominating Set.

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Idea: Find a bounded mim-width preserving reduction from SEMITOTAL DOMINATING SET to TOTAL DOMINATING SET.

- Let G be the input graph and (T, δ) the given branch decomposition with $w = \mathrm{mimw}_G(T, \delta)$.

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- Let G be the input graph and (T, δ) the given branch decomposition with $w = \mathrm{mimw}_G(T, \delta)$.
- Compute an appropriate G' such that:
  - $\gamma_{t2}(G) = \gamma_t(G')$ and a min semi-TD-set of G can be obtained from a min TD-set of G' in linear time.
  - G' has a branch decomposition (T', δ') with $\mathrm{mimw}_{G'}(T', \delta') \leq 2w$.
  - Both G' and (T', δ') can be computed in $O(n^3)$ time.

From semitotal to total

Theorem (Galby, M., Ries 2020)

There is an algorithm that, given a graph G and a branch decomposition (T, δ) for G with $w = \mathrm{mimw}_G(T, \delta)$, solves Semitotal Dominating Set *in $O(n^{4+6w})$ time.*

Idea: Find a bounded mim-width preserving reduction from Semitotal Dominating Set to Total Dominating Set.

- Let G be the input graph and (T, δ) the given branch decomposition with $w = \mathrm{mimw}_G(T, \delta)$.
- Compute an appropriate G' such that:
  - $\gamma_{t2}(G) = \gamma_t(G')$ and a min semi-TD-set of G can be obtained from a min TD-set of G' in linear time.
  - G' has a branch decomposition (T', δ') with $\mathrm{mimw}_{G'}(T', \delta') \leq 2w$.
  - Both G' and (T', δ') can be computed in $O(n^3)$ time.
- Find a min TD-set of G' with given branch decomposition (T', δ') by (Bui-Xuan et al. 2013).

From semitotal to total: $\gamma_{t2}(G) = \gamma_t(G')$

$\Longrightarrow$

$G = (V, E)$

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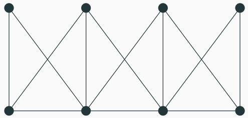
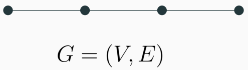


true twins

From semitotal to total: $\gamma_{t2}(G) = \gamma_t(G')$



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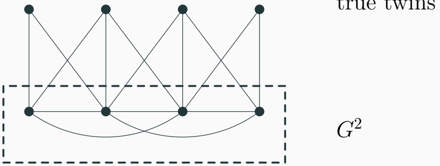

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$G = (V, E)$

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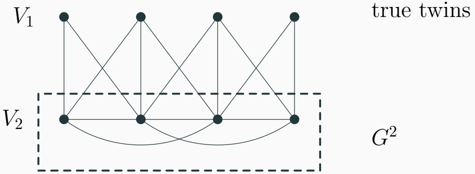
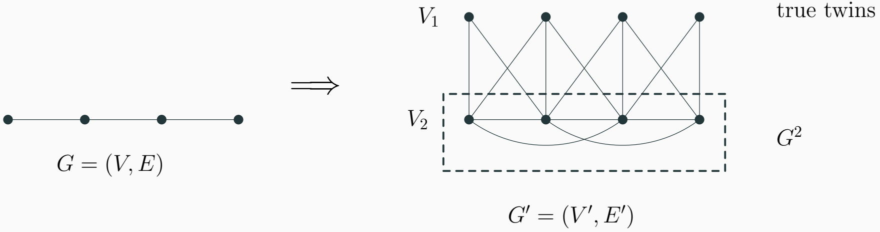


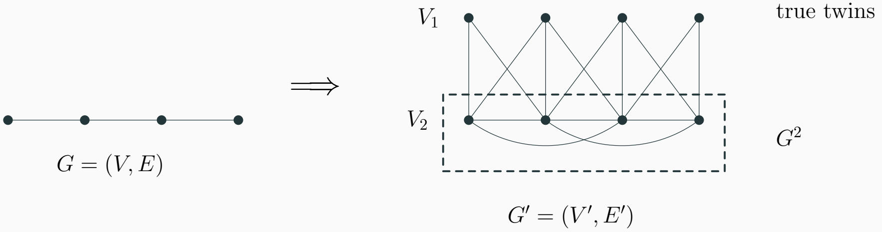
$G' = (V', E')$

From semitotal to total: $\gamma_{t2}(G) = \gamma_t(G')$



$\gamma_t(G') \leq \gamma_{t2}(G)$: If S is a semi-TD-set of G, then $S' = \{v_2 \in V_2 : v \in S\}$ is a TD-set of G'.

From semitotal to total: $\gamma_{t2}(G) = \gamma_t(G')$



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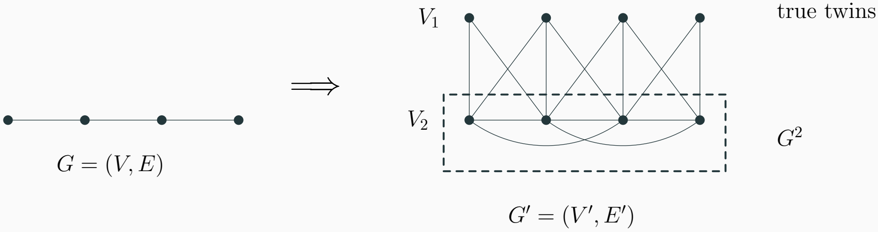

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- $S = \{v \in V : \{v_1, v_2\} \cap S' \neq \varnothing\}$ is a semi-TD-set of G.

From semitotal to total: if G has bounded mim-width, then G' has

Lemma (Galby, M., Ries 2020)

For any graph G not isomorphic to tK_1, $\mathrm{mimw}(G') \leq 2 \cdot \mathrm{mimw}(G)$.

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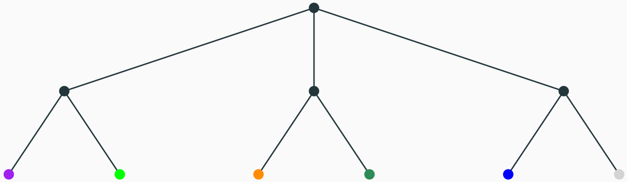

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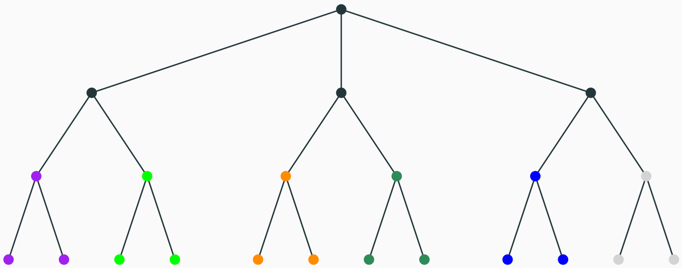
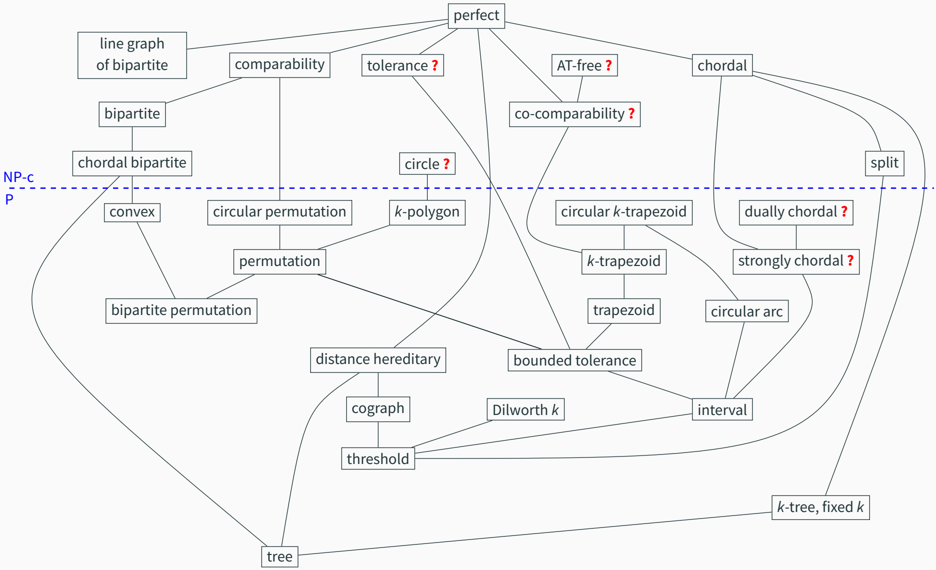


A broader view

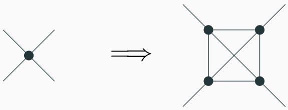


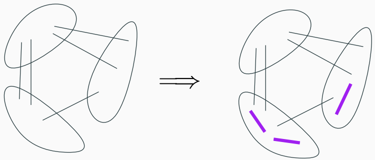
Outline

Behavior wrt graph operations

Lemma (Vatshelle 2012, BHMPP 2020+)

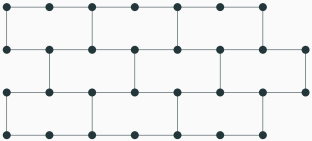
- ***Vertex deletion:*** $\mathrm{mimw}(G) - 1 \leq \mathrm{mimw}(G - v) \leq \mathrm{mimw}(G)$.
- ***1-subdivision of*** $e \in E(G)$***:*** $\mathrm{mimw}(G) \leq \mathrm{mimw}(G') \leq \mathrm{mimw}(G) + 1$.
- ***Clique implant on*** $v \in V(G)$***:*** $\mathrm{mimw}(G) \leq \mathrm{mimw}(G') \leq \mathrm{mimw}(G) + d(v)$.
- k***-partite partial complementation:*** $\mathrm{mimw}(G') \geq \frac{1}{k(k-1)} \cdot \mathrm{mimw}(G)$.
- ***Blocks:*** $\mathrm{mimw}(G) = \max\{\mathrm{mimw}(H) : H \text{ is a block of } G\}$. *Moreover, given branch decompositions of each block of G with mim-width at most k, we can compute a branch decomposition for G with mim-width at most k in poly time.*





Theorem (BHMPP 2020+)

Let W be an elementary $(n \times n)$-wall with $n \geq 7$. Then $\mathrm{mimw}(W) \geq \frac{\sqrt{n}}{30}$.



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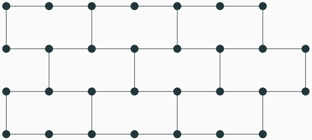

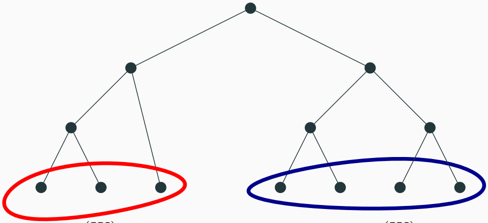

$\geq \frac{n(W)}{3}$ $\qquad$ $\geq \frac{n(W)}{3}$

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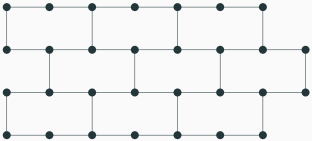
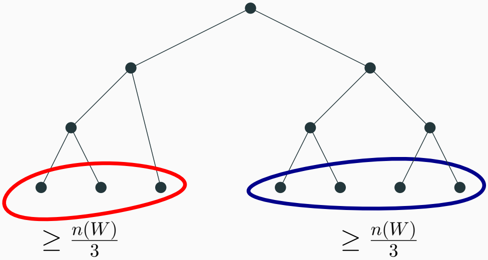
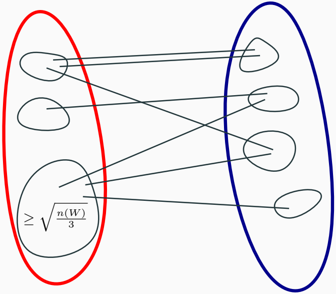
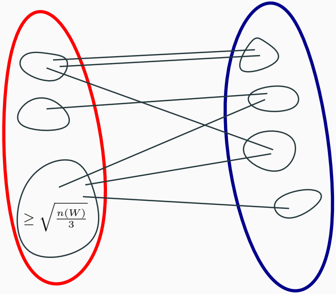




If $G[A, \overline{A}]$ is d-degenerate and has matching of size m, then $G[A, \overline{A}]$ has induced matching of size $m/(d+1)$ (Vatshelle 2012)


$\geq \sqrt{\frac{n(W)}{3}}$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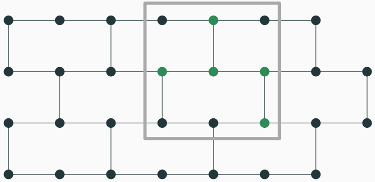

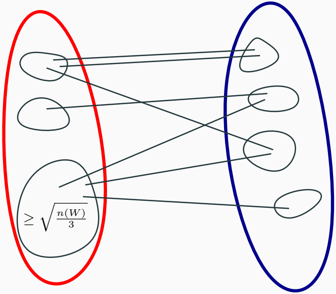
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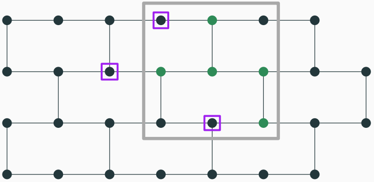
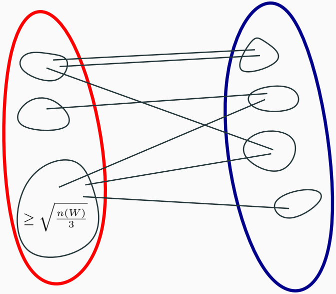
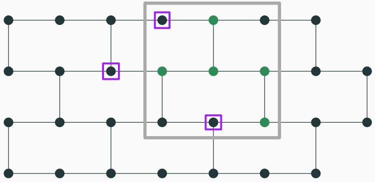
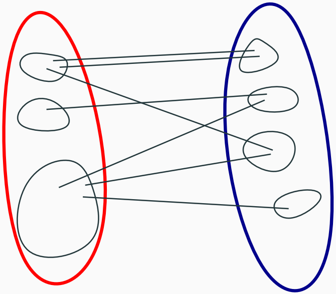
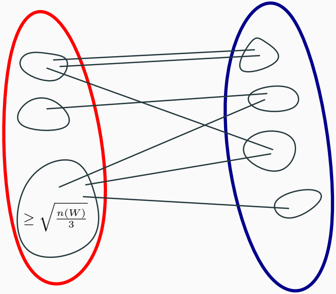


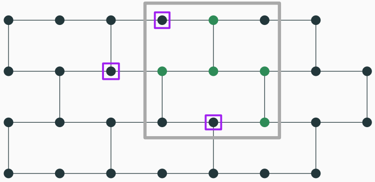




$$k > \sqrt{\frac{n(W)}{3}} \text{ comp.}$$







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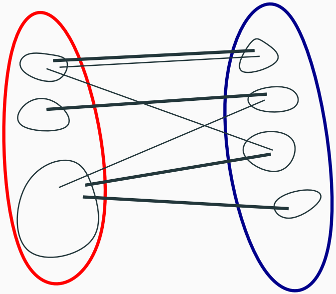
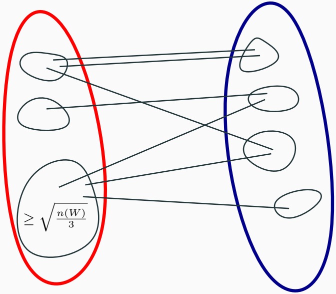
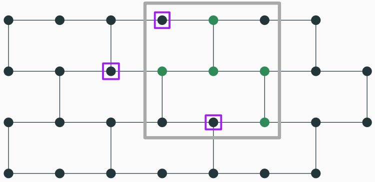
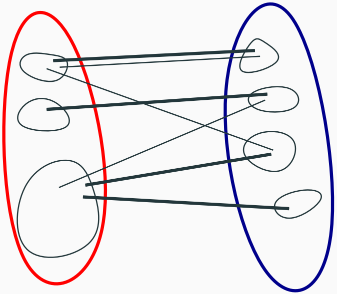






$k > \sqrt{\frac{n(W)}{3}}$ comp.



H: subgraph of $W[A_e, \overline{A_e}]$ induced by bold edges

$\rightsquigarrow$ Each component of H has size $n_i \geq 2$ and a matching of size $\geq (n_i - 1)/3 \geq n_i/6$ (Biedl et al. 2004)

$\rightsquigarrow$ H has a matching of size

$$\sum_{i=1}^{\ell} \frac{n_i}{6} = \frac{|V(H)|}{6} \geq \frac{k}{6} \geq \frac{1}{6} \cdot \sqrt{\frac{n(W)}{3}} \geq \frac{\sqrt{n}}{10}$$

The following graph classes have unbounded mim-width:

- Co-bipartite (Mengel 2018)
- Split (Mengel 2018)
- Strongly chordal (Mengel 2018)
- Chordal bipartite (Brault-Baron et al. 2015)
- Circle (Kang et al. 2017)
- Co-comparability (Kang et al. 2017)

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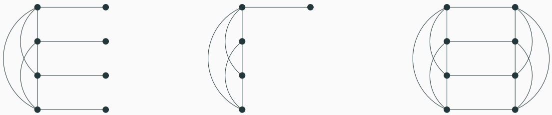
Theorem (BHMPP 2020+)

The class of H-free graphs has bounded mim-width if and only if $H \subseteq_i P_4$.

- If $H \subseteq_i P_4$, then H-free graphs are P_4-free and so have clique-width at most 2 and hence mim-width at most 2.
- Suppose H is such that the class of H-free graphs has bounded mim-width.
  $\rightsquigarrow$ H is a $(3P_1, 2P_2)$-free forest.
  $\rightsquigarrow$ $H \subseteq_i P_4$.

(H_1, H_2)-free graphs: New bounded cases

- Let $r \geq 3$. For any $(K_r \boxminus rP_1, 2P_2)$-free graph G and any $X \subseteq V(G)$, $\mathrm{cutmim}_G(X, \overline{X}) < \max\{6, r\}$.
- Let $r \geq 1$ and $t \geq 1$. For any $(K_r \boxminus P_1, tP_2)$-free graph G and any $X \subseteq V(G)$, $\mathrm{cutmim}_G(X, \overline{X}) < R(r, R(r, t))$.
- Let $r \geq 1$ and $s \geq 0$. For any $(K_r \boxminus K_r, sP_1 + P_2)$-free graph G and any $X \subseteq V(G)$, $\mathrm{cutmim}_G(X, \overline{X}) < R(R(r, s+1), s+1)$.
- If G is $(2P_2, K_{1,3})$-free, then $\mathrm{mimw}(G) < 6$ and we can construct in polynomial time a branch decomposition (T, δ) for G with $\mathrm{mimw}_G(T, \delta) < 6$.
- If G is $(2P_1 + P_2, \text{bowtie})$-free, then $\mathrm{mimw}(G) < R(14, 3)$ and we can construct in polynomial time a branch decomposition (T, δ) for G with $\mathrm{mimw}_G(T, \delta) < R(14, 3)$.



Complementation does not preserve mim-width: $(4P_1, 2P_2)$-free graphs have bounded mim-width but (K_4, C_4)-free graphs have unbounded mim-width.

Partial picture

Theorem (BHMPP 2020+)

Dichotomy when H_1 and H_2 are such that $|V(H_1)| + |V(H_2)| \leq 8$.

Theorem (BHMPP 2020+)

Let H_1 and H_2 be forests. Dichotomy except for:

1. $H_1 = 2P_2$ *and* $H_2 = K_{1,3} + sP_1$ *for* $s \geq 1$;
2. $H_1 = 2P_2$ *and* $H_2 = S_{1,1,2} + sP_1$ *for* $s \geq 0$.

Theorem (BHMPP 2020+)

Let H_1 and H_2 be connected graphs. Dichotomy except for:

1. $H_1 = P_5$ *and* $H_2 = \overline{S_{1,1,2}}$ *or* $\overline{K_{1,r} + sP_1}$ *for* $r \geq 3$ *and* $s \in \{1, 2\}$;
2. $H_1 = P_7$ *or* $S_{h,i,j}$ *for* $h \leq i \leq j \leq 4$ *with* $i + j \leq 6 \leq h + i + j$ *and* $H_2 = C_3$ *or* paw;
3. $H_1 = K_{1,3}$ *or* $S_{1,1,2}$ *and* $H_2 = \text{hammer}$.

More open problems

- Characterize graphs of mim-width at most 1.
- Further extend mim-width $\rightsquigarrow$ sim-width. Any problem poly-time solvable on bounded sim-width graphs?

THE GENERALIZED LOCALLY CHECKABLE PROBLEM IN BOUNDED TREEWIDTH GRAPHS

FLAVIA BONOMO-BRABERMAN AND CAROLINA LUCÍA GONZALEZ

ABSTRACT. We introduce a new problem that generalizes some previous attempts of covering locally checkable problems under the same umbrella. Optimization and decision problems such as $\{k\}$-dominating set, b-coloring, acyclic coloring and connected dominating set, can be seen as instances of this new problem.

We prove that this new problem can be solved, under mild conditions, in polynomial time for bounded treewidth graphs. As a consequence, we obtain polynomial-time algorithms to solve, for bounded treewidth graphs, Grundy domination and double Roman domination, among other problems for which no such algorithm was previously known. Moreover, by proving that (fixed) powers of bounded degree and bounded treewidth graphs are also bounded degree and bounded treewidth graphs, we can enlarge the family of problems that can be solved in polynomial time for these graph classes, including distance coloring problems and distance domination problems (for bounded distances).

Thank you!