On chordal and dually chordal graphs and their tree representations

Pablo De Caria

CONICET/ Departamento de Matemática, Universidad Nacional de La Plata

Koper, Slovenia, October 2014

Chordal graphs

Definition 1

A *chord* is an edge connecting two nonconsecutive vertices of a cycle.

Definition 2

A graph G is **chordal** if every cycle of length four or more in G has a chord.

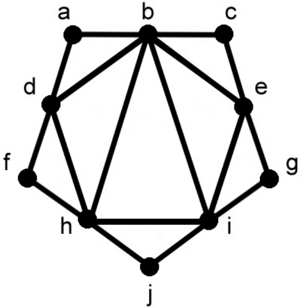
Chordal graphs

Definition 1

A *chord* is an edge connecting two nonconsecutive vertices of a cycle.

Definition 2

A graph G is **chordal** if every cycle of length four or more in G has a chord.



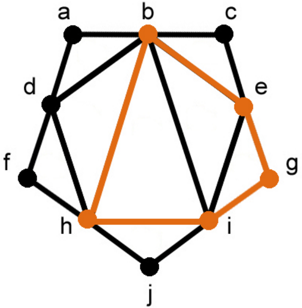
Chordal graphs

Definition 1

A *chord* is an edge connecting two nonconsecutive vertices of a cycle.

Definition 2

A graph G is **chordal** if every cycle of length four or more in G has a chord.



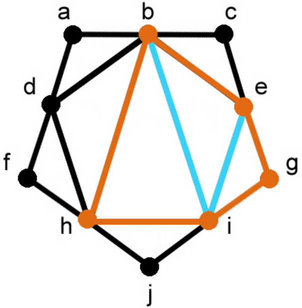
Chordal graphs

Definition 1

A *chord* is an edge connecting two nonconsecutive vertices of a cycle.

Definition 2

A graph G is **chordal** if every cycle of length four or more in G has a chord.



Simplicial vertex

Its neighborhood is a clique (maximal set of pairwise adjacent vertices).

Perfect elimination ordering

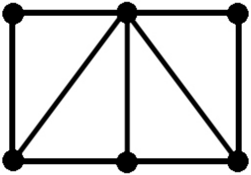
An ordering $v_1 v_2 ... v_n$ of the vertices of the graph such that v_i is simplicial in $G[v_i, ..., v_n]$ for all i, $1 \leq i \leq n$.

Simplicial vertex

Its neighborhood is a clique (maximal set of pairwise adjacent vertices).

Perfect elimination ordering

An ordering $v_1 v_2 ... v_n$ of the vertices of the graph such that v_i is simplicial in $G[v_i, ..., v_n]$ for all i, $1 \leq i \leq n$.

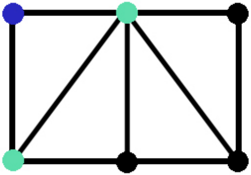


Simplicial vertex

Its neighborhood is a clique (maximal set of pairwise adjacent vertices).

Perfect elimination ordering

An ordering $v_1 v_2 ... v_n$ of the vertices of the graph such that v_i is simplicial in $G[v_i, ..., v_n]$ for all i, $1 \leq i \leq n$.

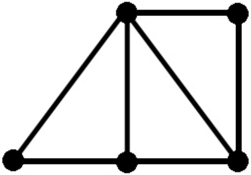


Simplicial vertex

Its neighborhood is a clique (maximal set of pairwise adjacent vertices).

Perfect elimination ordering

An ordering $v_1 v_2 ... v_n$ of the vertices of the graph such that v_i is simplicial in $G[v_i, ..., v_n]$ for all i, $1 \leq i \leq n$.

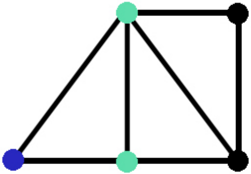


Simplicial vertex

Its neighborhood is a clique (maximal set of pairwise adjacent vertices).

Perfect elimination ordering

An ordering $v_1 v_2 ... v_n$ of the vertices of the graph such that v_i is simplicial in $G[v_i, ..., v_n]$ for all i, $1 \leq i \leq n$.

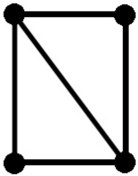


Simplicial vertex

Its neighborhood is a clique (maximal set of pairwise adjacent vertices).

Perfect elimination ordering

An ordering $v_1 v_2 ... v_n$ of the vertices of the graph such that v_i is simplicial in $G[v_i, ..., v_n]$ for all i, $1 \leq i \leq n$.

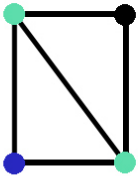


Simplicial vertex

Its neighborhood is a clique (maximal set of pairwise adjacent vertices).

Perfect elimination ordering

An ordering $v_1 v_2 ... v_n$ of the vertices of the graph such that v_i is simplicial in $G[v_i, ..., v_n]$ for all i, $1 \leq i \leq n$.

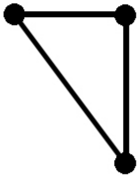


Simplicial vertex

Its neighborhood is a clique (maximal set of pairwise adjacent vertices).

Perfect elimination ordering

An ordering $v_1 v_2 ... v_n$ of the vertices of the graph such that v_i is simplicial in $G[v_i, ..., v_n]$ for all i, $1 \leq i \leq n$.

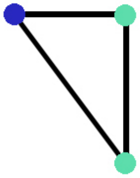


Simplicial vertex

Its neighborhood is a clique (maximal set of pairwise adjacent vertices).

Perfect elimination ordering

An ordering $v_1 v_2 \ldots v_n$ of the vertices of the graph such that v_i is simplicial in $G[v_i, \ldots, v_n]$ for all i, $1 \leq i \leq n$.



Simplicial vertex

Its neighborhood is a clique (maximal set of pairwise adjacent vertices).

Perfect elimination ordering

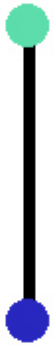
An ordering $v_1 v_2 \ldots v_n$ of the vertices of the graph such that v_i is simplicial in $G[v_i, \ldots, v_n]$ for all i, $1 \leq i \leq n$.

Simplicial vertex

Its neighborhood is a clique (maximal set of pairwise adjacent vertices).

Perfect elimination ordering

An ordering $v_1 v_2 ... v_n$ of the vertices of the graph such that v_i is simplicial in $G[v_i, ..., v_n]$ for all i, $1 \leq i \leq n$.



Simplicial vertex

Its neighborhood is a clique (maximal set of pairwise adjacent vertices).

Perfect elimination ordering

An ordering $v_1 v_2 ... v_n$ of the vertices of the graph such that v_i is simplicial in $G[v_i, ..., v_n]$ for all i, $1 \leq i \leq n$.

Simplicial vertex

Its neighborhood is a clique (maximal set of pairwise adjacent vertices).

Perfect elimination ordering

An ordering $v_1 v_2 ... v_n$ of the vertices of the graph such that v_i is simplicial in $G[v_i, ..., v_n]$ for all i, $1 \leq i \leq n$.

Simplicial vertex

Its neighborhood is a clique (maximal set of pairwise adjacent vertices).

Perfect elimination ordering

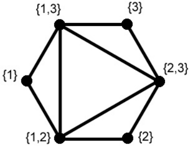
An ordering $v_1 v_2 ... v_n$ of the vertices of the graph such that v_i is simplicial in $G[v_i, ..., v_n]$ for all i, $1 \leq i \leq n$.

Theorem

A graph G is chordal if and only if it has a perfect elimination ordering.

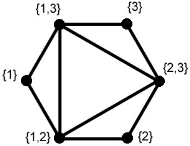
Intersection graph

The *intersection graph* of the family $\mathcal{F}$, or $L(\mathcal{F})$, has $\mathcal{F}$ as vertex set and F_1 and F_2 are adjacent in $L(\mathcal{F})$ if and only if $F_1 \cap F_2 \neq \emptyset$.



Intersection graph

The *intersection graph* of the family $\mathcal{F}$, or $L(\mathcal{F})$, has $\mathcal{F}$ as vertex set and F_1 and F_2 are adjacent in $L(\mathcal{F})$ if and only if $F_1 \cap F_2 \neq \emptyset$.

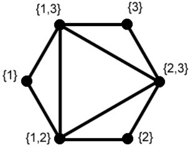


Another characterization

A graph is chordal if and only if it is the intersection graph of a family of subtrees of a tree.

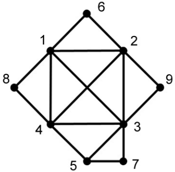
Intersection graph

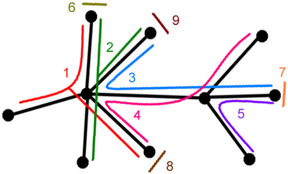
The *intersection graph* of the family $\mathcal{F}$, or $L(\mathcal{F})$, has $\mathcal{F}$ as vertex set and F_1 and F_2 are adjacent in $L(\mathcal{F})$ if and only if $F_1 \cap F_2 \neq \emptyset$.

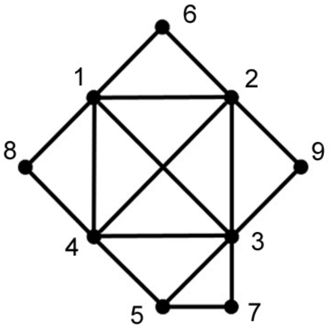


Another characterization

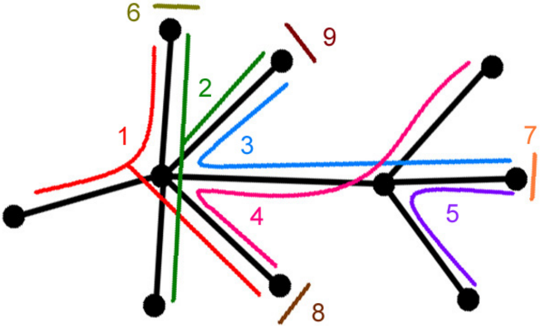
A graph is chordal if and only if it is the intersection graph of a family of subtrees of a tree.



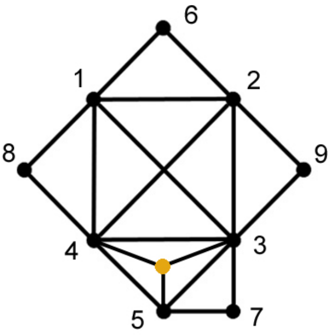


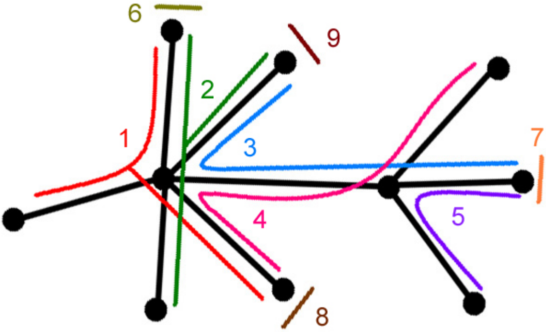


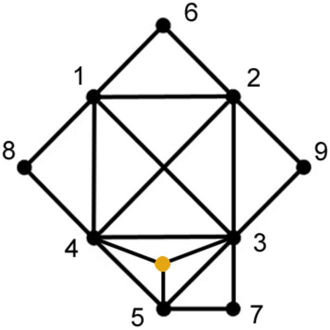
6
1
2
8
9
4
3
5
7

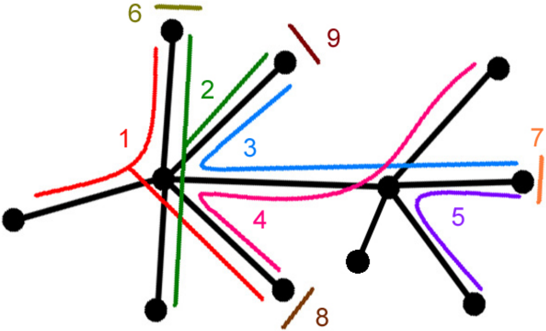


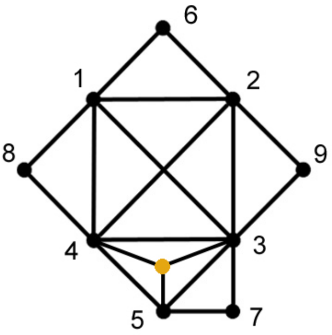
6
9
2
1
3
7
4
5
8

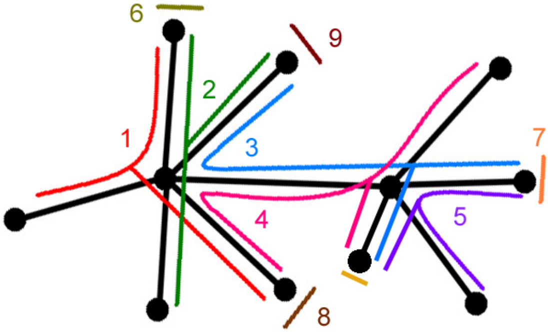

6
1
2
8
9
4
3
5
7


6
9
2
1
3
7
4
5
8


6
1
2
8
9
4
3
5
7


6
9
2
1
3
7
4
5
8


6
1
2
8
9
4
3
5
7


6
9
2
1
3
7
4
5
8

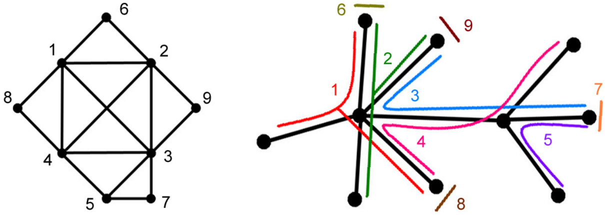
There are infinitely many representations for a chordal graph.

There are infinitely many representations for a chordal graph.

We can reduce the number of vertices in the representation by contracting every edge uv of the tree such that every subtree containing u also contains v.

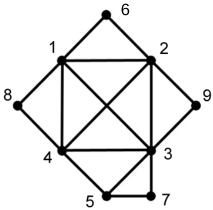
There are infinitely many representations for a chordal graph.

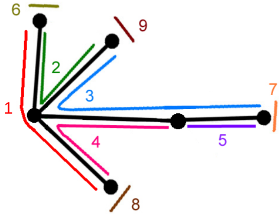
We can reduce the number of vertices in the representation by contracting every edge uv of the tree such that every subtree containing u also contains v.



There are infinitely many representations for a chordal graph.

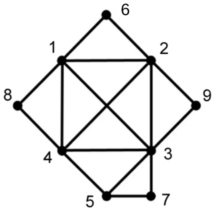
We can reduce the number of vertices in the representation by contracting every edge *uv* of the tree such that every subtree containing *u* also contains *v*.

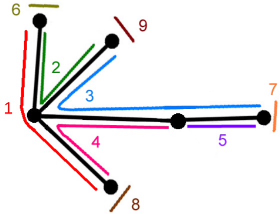




There are infinitely many representations for a chordal graph.

We can reduce the number of vertices in the representation by contracting every edge uv of the tree such that every subtree containing u also contains v.





When no more contractions are possible, the family $\mathcal{F}_v$, for $v \in V(T)$, of subtrees that contain v corresponds to pairwise adjacent vertices of the graph and no $\mathcal{F}_v$ contains another.

Conclusion

There is a correspondence between the vertices of T and the cliques of G.

The clique tree

A clique tree of G is a tree T whose vertices are the cliques of G and such that, for every $v \in V(G)$, the set $\mathcal{C}_v$ of cliques containing v induces a subtree in T.

The clique tree

A clique tree of G is a tree T whose vertices are the cliques of G and such that, for every $v \in V(G)$, the set $\mathcal{C}_v$ of cliques containing v induces a subtree in T.

Characterization

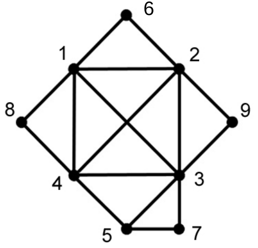
A graph is chordal if and only if it has a clique tree.

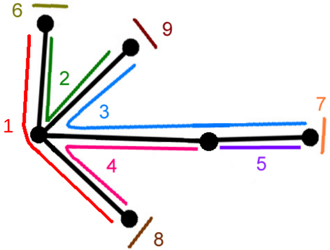
The clique tree

A clique tree of G is a tree T whose vertices are the cliques of G and such that, for every $v \in V(G)$, the set $\mathcal{C}_v$ of cliques containing v induces a subtree in T.

Characterization

A graph is chordal if and only if it has a clique tree.



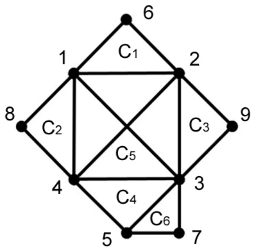


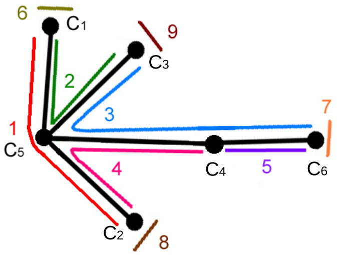
The clique tree

A clique tree of G is a tree T whose vertices are the cliques of G and such that, for every $v \in V(G)$, the set $\mathcal{C}_v$ of cliques containing v induces a subtree in T.

Characterization

A graph is chordal if and only if it has a clique tree.





Clique graphs

The *clique graph* of a graph G, denoted by $K(G)$, is the intersection graph of the cliques of G.

Property

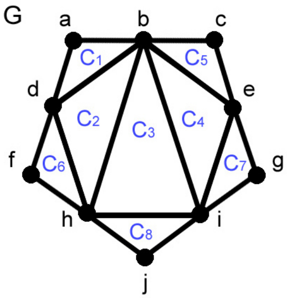
For G chordal and connected, any clique tree of G is a spanning tree of $K(G)$.

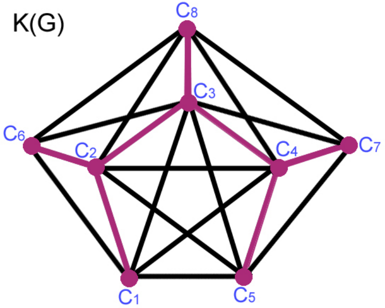
Clique graphs

The *clique graph* of a graph G, denoted by $K(G)$, is the intersection graph of the cliques of G.

Property

For G chordal and connected, any clique tree of G is a spanning tree of $K(G)$.



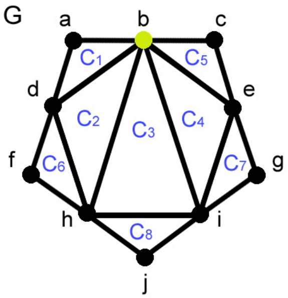


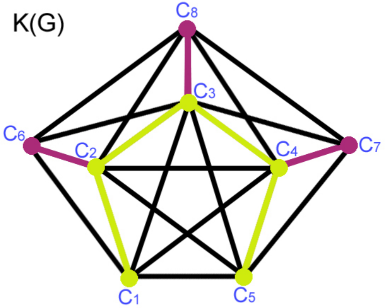
Clique graphs

The *clique graph* of a graph G, denoted by $K(G)$, is the intersection graph of the cliques of G.

Property

For G chordal and connected, any clique tree of G is a spanning tree of $K(G)$.



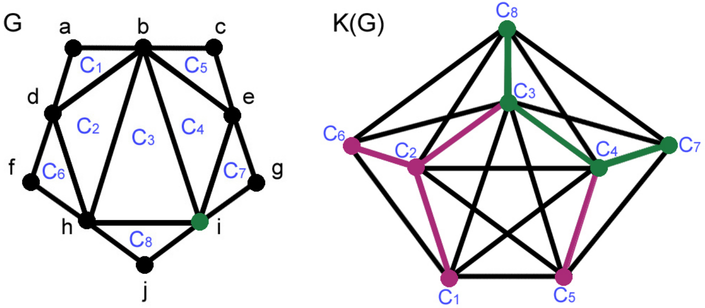


Clique graphs

The *clique graph* of a graph G, denoted by $K(G)$, is the intersection graph of the cliques of G.

Property

For G chordal and connected, any clique tree of G is a spanning tree of $K(G)$.

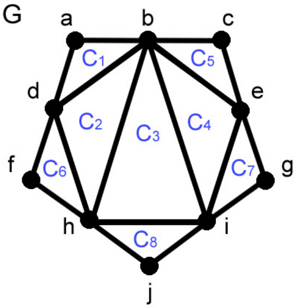


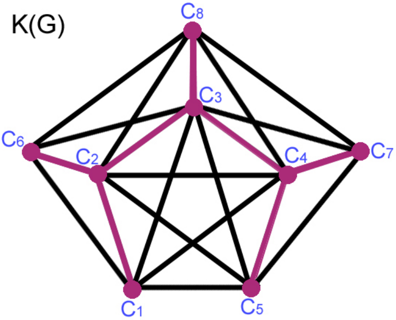
Property

Let T be a clique tree and C be a clique of G. Then the cliques of G that intersect C induce a subtree in T.

Property

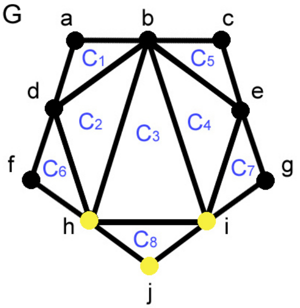
Let T be a clique tree and C be a clique of G. Then the cliques of G that intersect C induce a subtree in T.

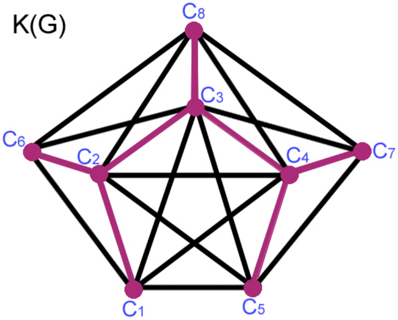




Property

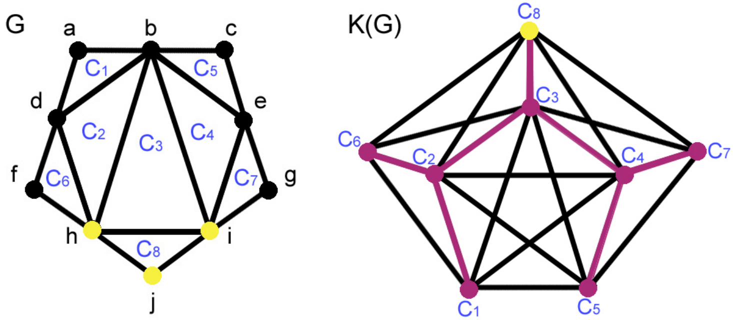
Let T be a clique tree and C be a clique of G. Then the cliques of G that intersect C induce a subtree in T.





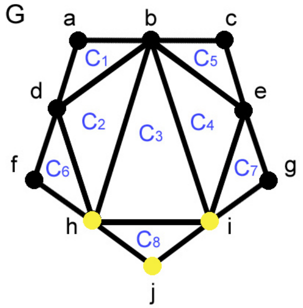
Property

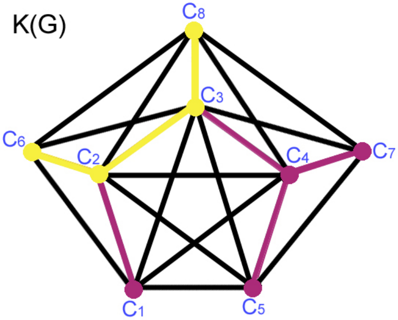
Let T be a clique tree and C be a clique of G. Then the cliques of G that intersect C induce a subtree in T.



Property

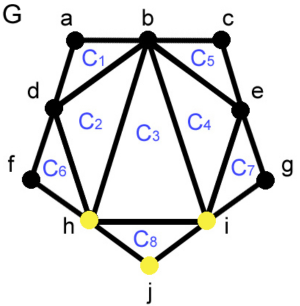
Let T be a clique tree and C be a clique of G. Then the cliques of G that intersect C induce a subtree in T.

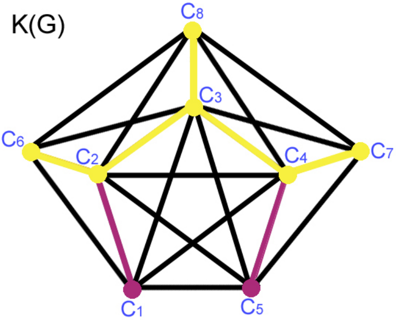




Property

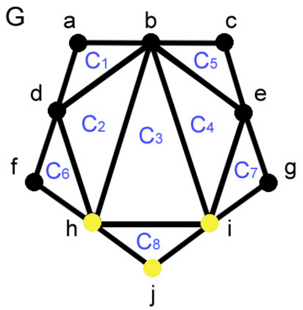
Let T be a clique tree and C be a clique of G. Then the cliques of G that intersect C induce a subtree in T.

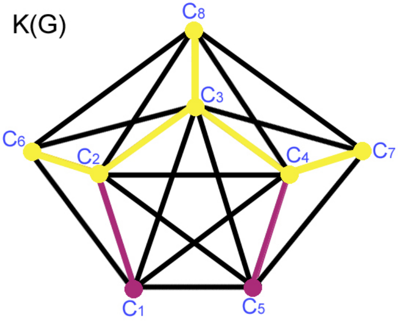




Property

Let T be a clique tree and C be a clique of G. Then the cliques of G that intersect C induce a subtree in T.





In other words, every clique tree of G verifies that every closed neighborhood of $K(G)$ induces a subtree.

Dually chordal graphs

A graph is dually chordal if it is the clique graph of a chordal graph.

Dually chordal graphs

A graph is dually chordal if it is the clique graph of a chordal graph.

Compatible tree

A compatible tree of a graph G is a tree T such that every closed neighborhood of G induces a subtree in T.

Dually chordal graphs

A graph is dually chordal if it is the clique graph of a chordal graph.

Compatible tree

A compatible tree of a graph G is a tree T such that every closed neighborhood of G induces a subtree in T.

Theorem

Every dually chordal graph has a compatible tree.

What is more, a graph is dually chordal if and only if it has a compatible tree.

Why dually chordal graphs?

Proposition

A tree T is compatible with G if and only if every clique induces a subtree in T.

Idea of proof: $C = \bigcap_{v \in C} N[v]$ and $N[v] = \bigcup_{v \in C} C$

Why dually chordal graphs?

Proposition

A tree T is compatible with G if and only if every clique induces a subtree in T.

Idea of proof: $C = \bigcap_{v \in C} N[v]$ and $N[v] = \bigcup_{v \in C} C$

Dual family

The dual of $\mathcal{F}$ is the family $D\mathcal{F} = \{D_v\}_{v \in \bigcup_{F \in \mathcal{F}} F}$, where

$D_v = \{F \in \mathcal{F} : \ v \in F\}$.

Why dually chordal graphs?

Proposition

A tree T is compatible with G if and only if every clique induces a subtree in T.

Idea of proof: $C = \bigcap_{v \in C} N[v]$ and $N[v] = \bigcup_{v \in C} C$

Dual family

The dual of $\mathcal{F}$ is the family $D\mathcal{F} = \{D_v\}_{v \in \bigcup_{F \in_F} F}$, where

$D_v = \{F \in \mathcal{F} : \ v \in F\}$.

Result

Define $\mathcal{C}(G)$ as the family of cliques of a graph G.

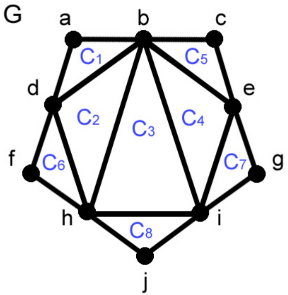
- G is dually chordal if and only if there exists a tree such that every member of $\mathcal{C}(G)$ induces a subtree.
- G is chordal if and only if there exists a tree such that every member of $D\mathcal{C}(G)$ induces a subtree.

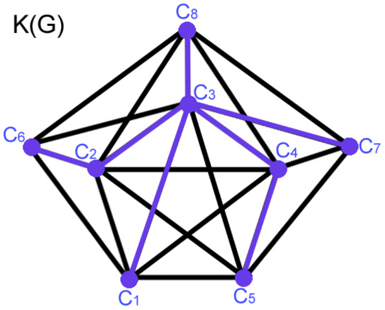
Property: Every clique tree of a graph G is a compatible tree of $K(G)$.

Is the converse true?

Property: Every clique tree of a graph G is a compatible tree of $K(G)$.

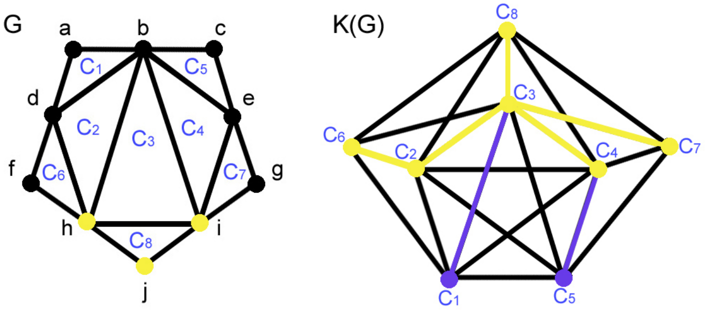
Is the converse true?





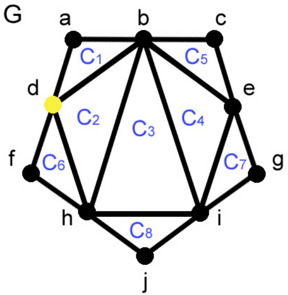
Property: Every clique tree of a graph G is a compatible tree of $K(G)$.

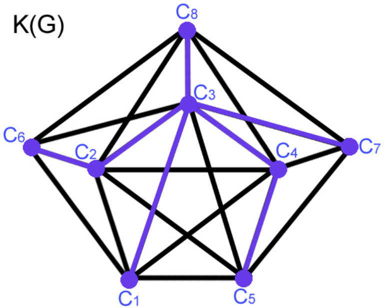
Is the converse true?



Property: Every clique tree of a graph G is a compatible tree of $K(G)$.

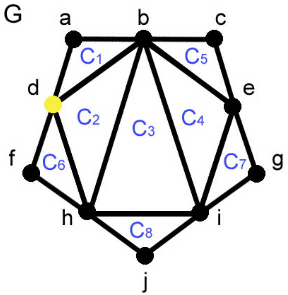
Is the converse true?

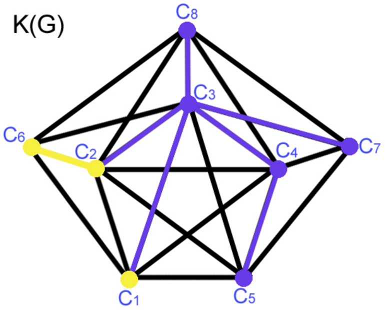




Property: Every clique tree of a graph G is a compatible tree of $K(G)$.

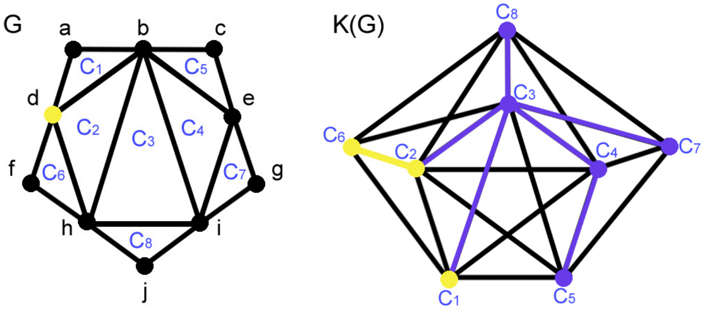
Is the converse true?





Property: Every clique tree of a graph G is a compatible tree of $K(G)$.

Is the converse true?



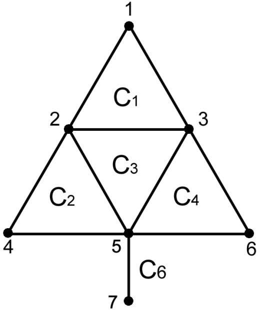
A chordal graph G is said to be **basic chordal** if the clique trees of G are exactly the compatible trees of $K(G)$.

G is *basic chordal* if the compatible trees of $K(G)$ are exactly the clique trees of G.

Let $\mathcal{S}(G)$ be the family of minimal vertex separators of G and let $S \in \mathcal{S}(G)$.

$\mathcal{C}_S$: Cliques that contain S

B_S: Consists of every C such that $C \cap D \neq \emptyset$ for $D \in \mathcal{C}(G)$ such that $D \cap S \neq \emptyset$.



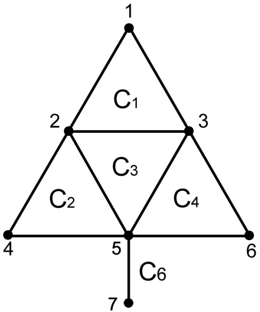
$$S = \{2, 5\}$$

G is *basic chordal* if the compatible trees of $K(G)$ are exactly the clique trees of G.

Let $\mathcal{S}(G)$ be the family of minimal vertex separators of G and let $S \in \mathcal{S}(G)$.

$\mathcal{C}_S$: Cliques that contain S

B_S: Consists of every C such that $C \cap D \neq \emptyset$ for $D \in \mathcal{C}(G)$ such that $D \cap S \neq \emptyset$.



$S = \{2, 5\}$

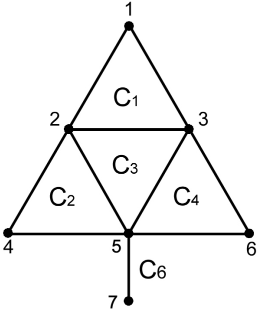
$\mathcal{C}_S = \{C_2, C_3\}$

G is *basic chordal* if the compatible trees of $K(G)$ are exactly the clique trees of G.

Let $\mathcal{S}(G)$ be the family of minimal vertex separators of G and let $S \in \mathcal{S}(G)$.

$\mathcal{C}_S$: Cliques that contain S

B_S: Consists of every C such that $C \cap D \neq \emptyset$ for $D \in \mathcal{C}(G)$ such that $D \cap S \neq \emptyset$.



$$S = \{2, 5\}$$

$$\mathcal{C}_S = \{C_2, C_3\}$$

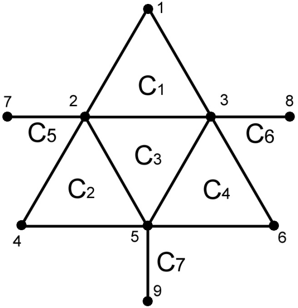
$$B_S = \{C_2, C_3, C_4\}$$

Theorem

A graph G is basic chordal *iff* $B_S = \mathcal{C}_S$ for all $S \in \mathcal{S}(G)$.

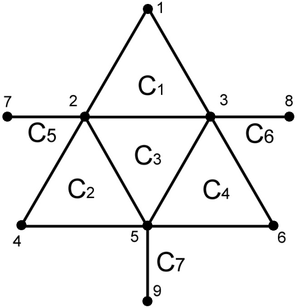
Theorem

A graph G is basic chordal *iff* $B_S = \mathcal{C}_S$ for all $S \in \mathcal{S}(G)$.



Theorem

A graph G is basic chordal *iff* $B_S = \mathcal{C}_S$ for all $S \in \mathcal{S}(G)$.



S	B_S	$\mathcal{C}_S$	$B_S = \mathcal{C}_S$
$\{2,3\}$	$\{C_1, C_3\}$	$\{C_1, C_3\}$	✓
$\{2,5\}$	$\{C_2, C_3\}$	$\{C_2, C_3\}$	✓
$\{3,5\}$	$\{C_3, C_4\}$	$\{C_3, C_4\}$	✓
$\{2\}$	$\{C_1, C_2, C_3, C_5\}$	$\{C_1, C_2, C_3, C_5\}$	✓
$\{3\}$	$\{C_1, C_3, C_4, C_6\}$	$\{C_1, C_3, C_4, C_6\}$	✓
$\{5\}$	$\{C_2, C_3, C_4, C_7\}$	$\{C_2, C_3, C_4, C_7\}$	✓

If G is chordal,

$\mathcal{SC}(G)$: Sets that induce a subtree of every clique tree of G.

Example: The members of $D\mathcal{C}(G)$.

If G is dually chordal,

$\mathcal{SDC}(G)$: Sets that induce a subtree of every compatible tree of G.

Example: Cliques, closed neighborhoods and minimal vertex separators.

If G is chordal,

$\mathcal{SC}(G)$: Sets that induce a subtree of every clique tree of G.

Example: The members of $D\mathcal{C}(G)$.

If G is dually chordal,

$\mathcal{SDC}(G)$: Sets that induce a subtree of every compatible tree of G.

Example: Cliques, closed neighborhoods and minimal vertex separators.

Theorem

A chordal graph G is basic chordal if and only if $\mathcal{SC}(G) = \mathcal{SDC}(K(G))$.

Connected unions

$\bigcup_{F \in \mathcal{F}} F$ is connected if $L(\mathcal{F})$ is connected.

Connected unions of members of $\mathcal{SC}(G)/\mathcal{SDC}(G)$ are in $\mathcal{SC}(G)/\mathcal{SDC}(G)$.

Connected unions

$\bigcup_{F \in \mathcal{F}} F$ is connected if $L(\mathcal{F})$ is connected.

Connected unions of members of $\mathcal{SC}(G)/\mathcal{SDC}(G)$ are in $\mathcal{SC}(G)/\mathcal{SDC}(G)$.

Families closed under connected unions are characterized by the existence of a basis.

A **basis** of $\mathcal{F}$ is a minimal subfamily $\mathcal{B}$ such that every $F \in \mathcal{F}$ with $|F| \geq 2$ can be expressed as the connected union of members of $\mathcal{B}$.

Connected unions

$\bigcup_{F \in \mathcal{F}} F$ is connected if $L(\mathcal{F})$ is connected.

Connected unions of members of $\mathcal{SC}(G)/\mathcal{SDC}(G)$ are in $\mathcal{SC}(G)/\mathcal{SDC}(G)$.

Families closed under connected unions are characterized by the existence of a basis.

A **basis** of $\mathcal{F}$ is a minimal subfamily $\mathcal{B}$ such that every $F \in \mathcal{F}$ with $|F| \geq 2$ can be expressed as the connected union of members of $\mathcal{B}$.

Example: $A = \{1, 2, 3, 4\}$

$\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, $\{2, 4\}$, $\{3, 4\}$ form the basis of the power set of A.

$\{1, 2, 3, 4\} = \{1, 2\} \cup \{2, 3\} \cup \{3, 4\}$

Connected unions

$\bigcup_{F \in \mathcal{F}} F$ is connected if $L(\mathcal{F})$ is connected.

Connected unions of members of $\mathcal{SC}(G)/\mathcal{SDC}(G)$ are in $\mathcal{SC}(G)/\mathcal{SDC}(G)$.

Families closed under connected unions are characterized by the existence of a basis.

A **basis** of $\mathcal{F}$ is a minimal subfamily $\mathcal{B}$ such that every $F \in \mathcal{F}$ with $|F| \geq 2$ can be expressed as the connected union of members of $\mathcal{B}$.

Example: $A = \{1, 2, 3, 4\}$

$\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, $\{2, 4\}$, $\{3, 4\}$ form the basis of the power set of A.

$\{1, 2, 3, 4\} = \{1, 2\} \cup \{2, 3\} \cup \{3, 4\}$

A basis is always unique and consists of the sets that cannot be expressed as connected unions of others.

Why basic chordal graphs?

G is basic chordal if and only if $\mathcal{SC}(G)$ and $\mathcal{SDC}(K(G))$ have the same basis.

Why basic chordal graphs?

G is basic chordal if and only if $\mathcal{SC}(G)$ and $\mathcal{SDC}(K(G))$ have the same basis.

What is the basis of $\mathcal{SC}(G)$ for G chordal?

The basis consists of the sets $\mathcal{C}_S$, $S \in \mathcal{S}(G)$.

What is the basis of $\mathcal{SDC}(K(G))$?

It consists of the sets B_S, $S \in \mathcal{S}(G)$.

Why basic chordal graphs?

G is basic chordal if and only if $\mathcal{SC}(G)$ and $\mathcal{SDC}(K(G))$ have the same basis.

What is the basis of $\mathcal{SC}(G)$ for G chordal?

The basis consists of the sets $\mathcal{C}_S$, $S \in \mathcal{S}(G)$.

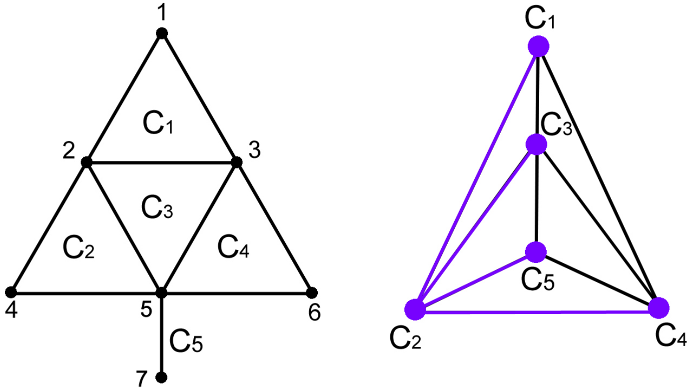
What is the basis of $\mathcal{SDC}(K(G))$?

It consists of the sets B_S, $S \in \mathcal{S}(G)$.

How can the basis for a dually chordal graph be obtained without using chordal graphs?

Take T compatible and, for every $uv \in E(T)$, Consider

$$\bigcap_{\{u,v\} \subseteq C} C = \bigcap_{\{u,v\} \subseteq N[w]} N[w].$$



The only clique containing C_1C_2, is $\{C_1, C_2, C_3, C_4\}$. But it is not a basic set for the graph at left.

Clique graphs of basic chordal graphs

Separating family

$\mathcal{F}$ is separating if for every vertex v, $\bigcap_{v \in F} F = \{v\}$.

Clique graphs of basic chordal graphs

Separating family

$\mathcal{F}$ is separating if for every vertex v, $\bigcap_{v \in F} F = \{v\}$.

Theorem

- $K(BASIC\ CHORDAL) = DUALLY\ CHORDAL$
- For G dually chordal, the basic chordal graphs with G as a clique graph are of the form $L(\mathcal{F})$, where $\mathcal{F}$ is separating, $\mathcal{F} \subseteq \mathcal{SDC}(G)$ and $\mathcal{F}$ covers all the edges of G.

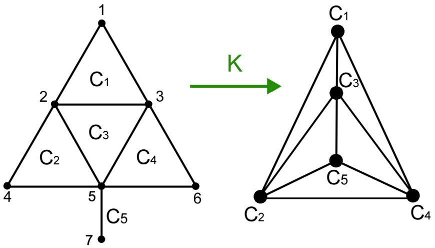
Clique graphs of basic chordal graphs

Separating family

$\mathcal{F}$ is separating if for every vertex v, $\bigcap_{v \in F} F = \{v\}$.

Theorem

- $K(BASIC\ CHORDAL) = DUALLY\ CHORDAL$
- For G dually chordal, the basic chordal graphs with G as a clique graph are of the form $L(\mathcal{F})$, where $\mathcal{F}$ is separating, $\mathcal{F} \subseteq \mathcal{SDC}(G)$ and $\mathcal{F}$ covers all the edges of G.



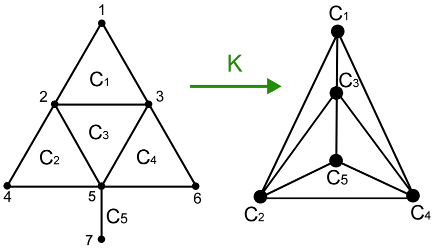
Clique graphs of basic chordal graphs

Separating family

$\mathcal{F}$ is separating if for every vertex v, $\bigcap_{v \in F} F = \{v\}$.

Theorem

- $K(BASIC\ CHORDAL) = DUALLY\ CHORDAL$
- For G dually chordal, the basic chordal graphs with G as a clique graph are of the form $L(\mathcal{F})$, where $\mathcal{F}$ is separating, $\mathcal{F} \subseteq \mathcal{SDC}(G)$ and $\mathcal{F}$ covers all the edges of G.



$A = \{C_1, C_2, C_3, C_4\}$, $B = \{C_2, C_3, C_4, C_5\}$, $C = \{C_1\}$, $D = \{C_2\}$, $E = \{C_3\}$, $F = \{C_4\}$, $G = \{C_5\}$

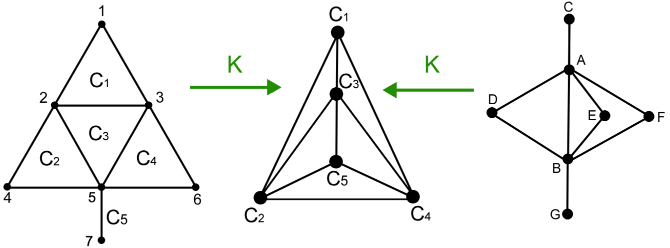
Clique graphs of basic chordal graphs

Separating family

$\mathcal{F}$ is separating if for every vertex v, $\bigcap_{v \in F} F = \{v\}$.

Theorem

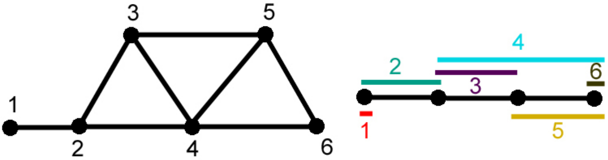
- $K(BASIC\ CHORDAL) = DUALLY\ CHORDAL$
- For G dually chordal, the basic chordal graphs with G as a clique graph are of the form $L(\mathcal{F})$, where $\mathcal{F}$ is separating, $\mathcal{F} \subseteq \mathcal{SDC}(G)$ and $\mathcal{F}$ covers all the edges of G.



$A = \{C_1, C_2, C_3, C_4\}$, $B = \{C_2, C_3, C_4, C_5\}$, $C = \{C_1\}$, $D = \{C_2\}$, $E = \{C_3\}$, $F = \{C_4\}$, $G = \{C_5\}$

Applications

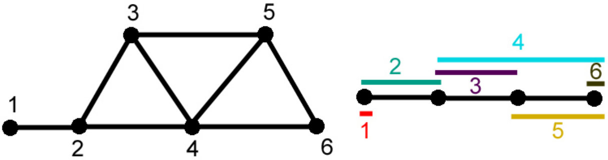
Leafage of a chordal graph G: It is the minimum number of leaves of a clique tree of G.



Can be found polinomially with an algorithm developed by M. Habib and J. Stacho.

Applications

Leafage of a chordal graph G: It is the minimum number of leaves of a clique tree of G.



Can be found polinomially with an algorithm developed by M. Habib and J. Stacho.

Dual leafage of a graph G: Minimum number of leaves of a compatible tree of G.

The dual leafage of G is equal to the leafage of any basic chordal graph H with $K(H) = G$.

Problem

Let G be a dually chordal graph, $|V(G)| \geq 3$ and $A \subseteq V(G)$. Is there a compatible tree of G that has A as its set of leaves?

Problem

Let G be a dually chordal graph, $|V(G)| \geq 3$ and $A \subseteq V(G)$. Is there a compatible tree of G that has A as its set of leaves?

Necessary condition

Every vertex of A is dominated by another in $V(G) \setminus A$.

Problem

Let G be a dually chordal graph, $|V(G)| \geq 3$ and $A \subseteq V(G)$. Is there a compatible tree of G that has A as its set of leaves?

Necessary condition

Every vertex of A is dominated by another in $V(G) \setminus A$.

Let G^* be obtained from G by adding, for each $v \in A$, the vertex v^* and the edge vv^*.

Theorem

If every vertex of A is dominated by another in $V(G) \setminus A$, then G has a compatible tree with set of leaves equal to A if and only if the dual leafage of G^* equals $|A|$.

Problem

Given a family $\mathcal{T}$ on a set V of vertices. Is there a chordal graph whose clique trees are exactly those of $\mathcal{T}$?

Problem

Given a family $\mathcal{T}$ on a set V of vertices. Is there a chordal graph whose clique trees are exactly those of $\mathcal{T}$?

Theorem

- T is a clique tree of G if and only if $\mathcal{C}_S$ induces a subtree of T for every $S \in \mathcal{S}(G)$.
- The intersection graph of $\{\mathcal{C}_S\}_{S \in \mathcal{S}(G)} \cup \{\{C\}\}_{C \in \mathcal{C}(G)}$ has the same clique trees as G.

Problem

Given a family $\mathcal{T}$ on a set V of vertices. Is there a chordal graph whose clique trees are exactly those of $\mathcal{T}$?

Theorem

- T is a clique tree of G if and only if $\mathcal{C}_S$ induces a subtree of T for every $S \in \mathcal{S}(G)$.
- The intersection graph of $\{\mathcal{C}_S\}_{S\in\mathcal{S}(G)} \cup \{\{C\}\}_{C\in\mathcal{C}(G)}$ has the same clique trees as G.

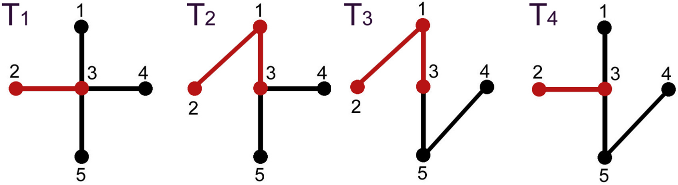
Question

How to identify the sets $\mathcal{C}_S$ when the graph is unknown and all what we know about it are its clique trees?

Definitions

$T[a, b]$: Vertices in the path of T from a to b.

$$\mathcal{T}[a, b] = \bigcup_{T \in \mathcal{T}} T[a, b].$$

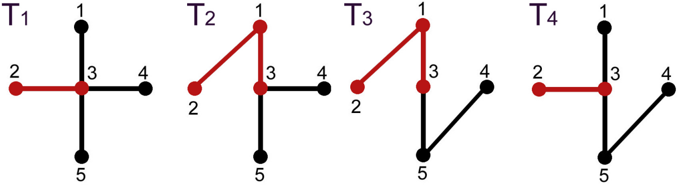


$\mathcal{T}[2, 3] = \{1,2,3\}$

Definitions

$T[a, b]$: Vertices in the path of T from a to b.

$$\mathcal{T}[a, b] = \bigcup_{T \in \mathcal{T}} T[a, b].$$



$\mathcal{T}[2, 3] = \{1,2,3\}$

Theorem

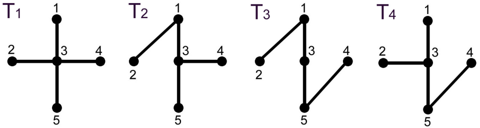
Let G be chordal, $\mathcal{T}$ be its family of clique trees and $T \in \mathcal{T}$. Then $\{\mathcal{C}_S\}_{S \in \mathcal{S}(G)} = \{\mathcal{T}[C, C'] : \; CC' \in E(T)\}$.

Theorem

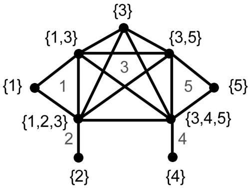
Let $\mathcal{T}$ be a family of trees on a set of vertices V, $T \in \mathcal{T}$ and $\mathcal{F} = \{\mathcal{T}[u,v]\}_{uv \in E(T)} \cup \{\{v\}\}_{v \in V(T)}$. Then $\mathcal{T}$ is the family of clique trees of a chordal graph if and only if $L(\mathcal{F})$ is chordal and and has $|\mathcal{T}|$ clique trees.

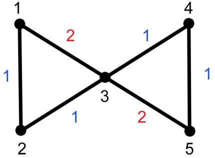
Theorem

Let $\mathcal{T}$ be a family of trees on a set of vertices V, $T \in \mathcal{T}$ and $\mathcal{F} = \{\mathcal{T}[u,v]\}_{uv \in E(T)} \cup \{\{v\}\}_{v \in V(T)}$. Then $\mathcal{T}$ is the family of clique trees of a chordal graph if and only if $L(\mathcal{F})$ is chordal and and has $|\mathcal{T}|$ clique trees.



$\mathcal{T}[1,3] = \{1,3\}$ $\mathcal{T}[2,3] = \{1,2,3\}$ $\mathcal{T}[3,4] = \{3,4,5\}$ $\mathcal{T}[3,5] = \{3,5\}$





- Every set of compatible trees of some dually chordal graph is the set of clique trees of some chordal graph.
- However, the converse is not true.
- Determining the complexity of recognizing families of compatible trees is an open problem.