The Price of Connectivity for Vertex Cover

Oliver Schaudt

Joint work with E. Camby, J. Cardinal and S. Fiorini (all from ULB)

Department for Computer Science
Group Faigle/Schrader
University of Cologne

May 2012

Basics

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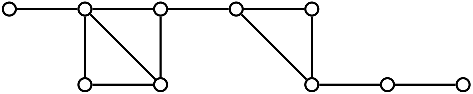

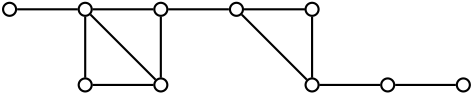

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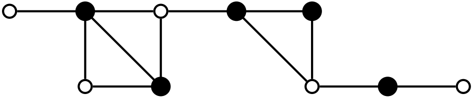

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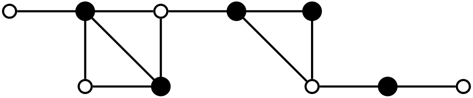

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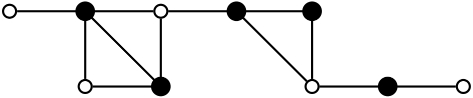
- The **vertex cover number** τ is the minimum size of a vertex cover.



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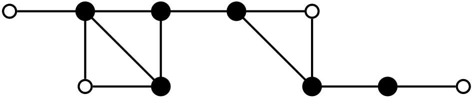

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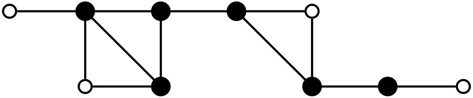

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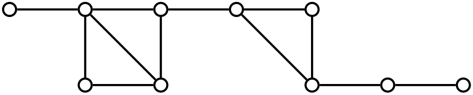

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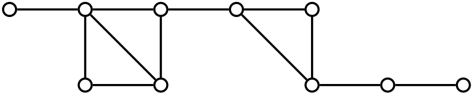

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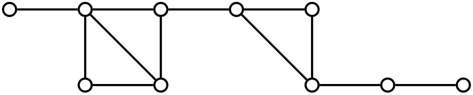

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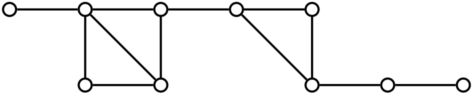

- Here, $\tau = 5$, $\tau_c = 6 \implies PoC = 6/5$.
- In general, $1 \leq PoC < 2$.

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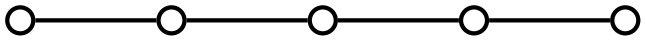

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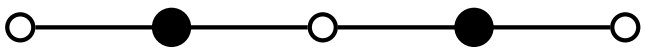

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- Thus P_5 is critical.

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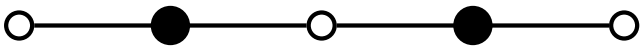

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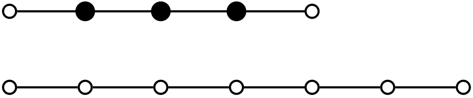

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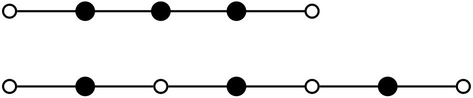

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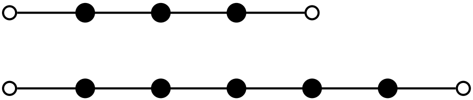

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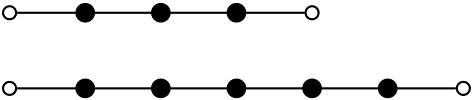

- Any path of even length ≥ 4 is critical.

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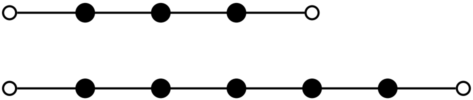

- Any path of even length ≥ 4 is critical.
- $\tau(P_{2k+1}) = k$, $\tau_c(P_{2k+1}) = 2k - 1$

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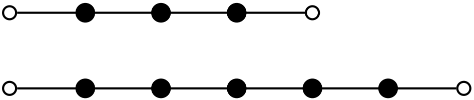

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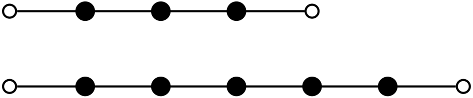

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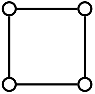

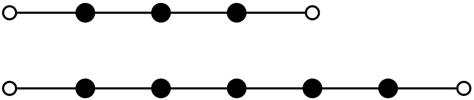

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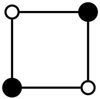

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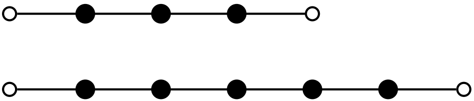

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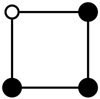

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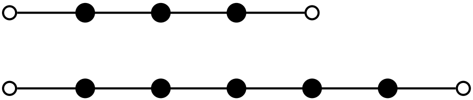

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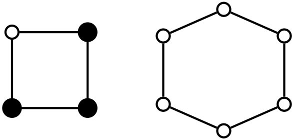

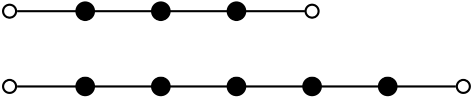

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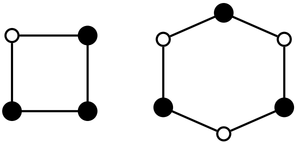

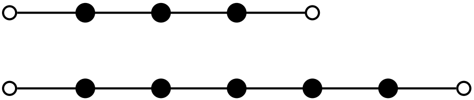

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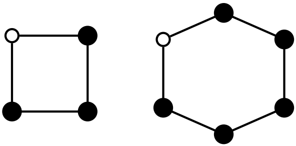

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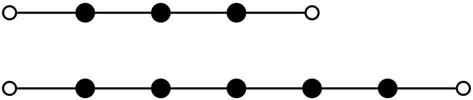

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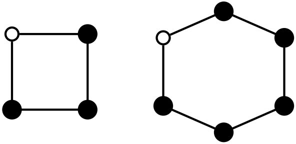

- Any cycle of even length is critical.

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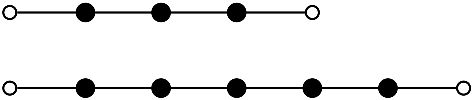

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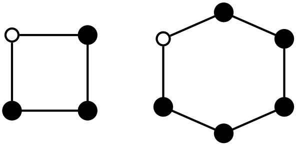

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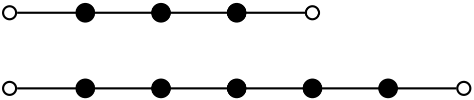

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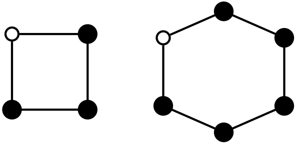

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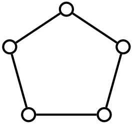

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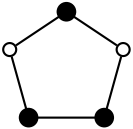

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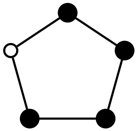

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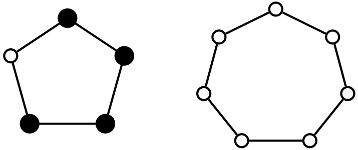

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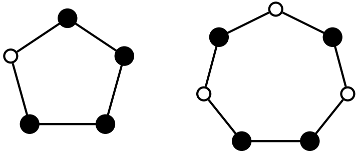

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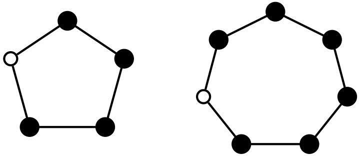

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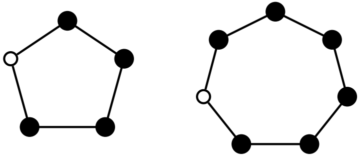

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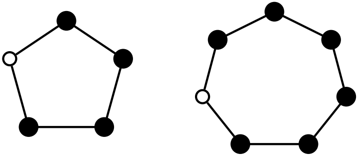

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- $PoC(C_{2k+1}) = 2 - 1/(k+1)$.

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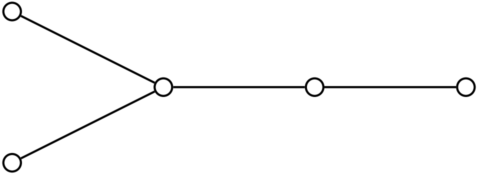

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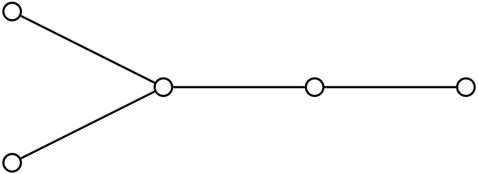

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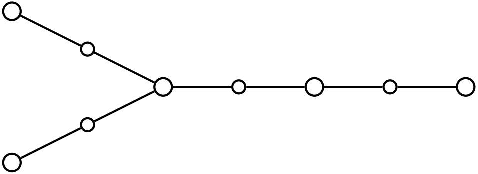

Critical trees

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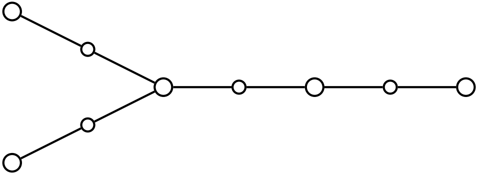
- Start with a tree.
- Subdivide each edge.
- Then attach a leaf to each leaf.



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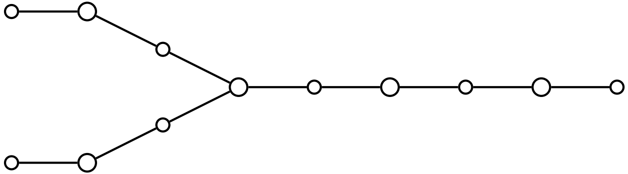

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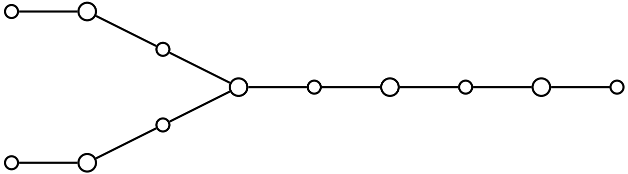

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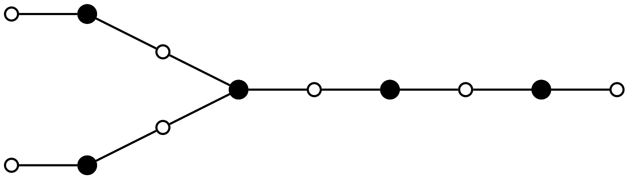

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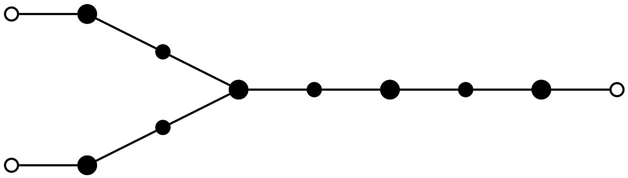
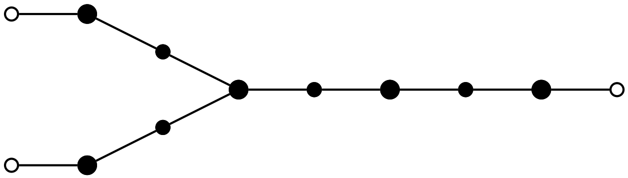


Critical trees

Special trees

- Start with a tree.
- Subdivide each edge.
- Then attach a leaf to each leaf.
- Such tree we call a **special tree**.



Observation

A tree is critical if and only if it is a special tree.

Critical chordal graphs

Critical chordal graphs

Theorem

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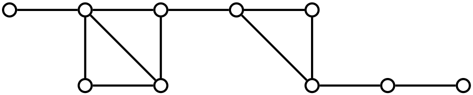

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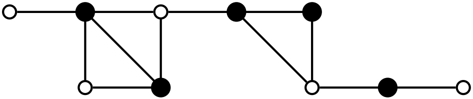

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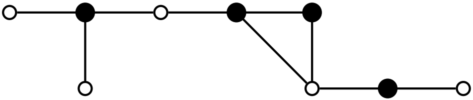

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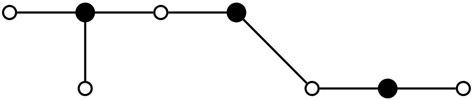

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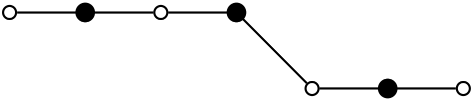

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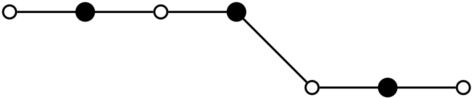

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PoC-perfect graphs

PoC-perfect graphs

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Let G be a graph. The following assertions are equivalent:

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Let G be a graph. The following assertions are equivalent:

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- G is (P_5, C_4, C_5)-free.

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Let G be a graph. The following assertions are equivalent:

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- G is chordal and P_5-free.

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- "$\Longrightarrow$":
  - $PoC(P_5) = 3/2$,

Theorem

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- "$\Longrightarrow$":
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- Proof:
- "$\Longrightarrow$":
  - $PoC(P_5) = 3/2$, $PoC(C_4) = 3/2$, and $PoC(C_5) = 4/3$.

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  - $PoC(P_5) = 3/2$, $PoC(C_4) = 3/2$, and $PoC(C_5) = 4/3$.
- "$\Longleftarrow$":
  - Let G be a chordal graph with $PoC(G) > 1$.

PoC-perfect graphs

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Let G be a graph. The following assertions are equivalent:

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  - Let G be a chordal graph with $PoC(G) > 1$.
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  - Thus G is a special tree.

PoC-perfect graphs

Theorem

Let G be a graph. The following assertions are equivalent:

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- "$\Longleftarrow$":
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  - We can assume that G is critical.
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PoC-near-perfect graphs

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Let G be a graph. The following assertions are equivalent:

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$\vdots$

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Thanks!