Path algebra and domination problems on graph products

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Polymers

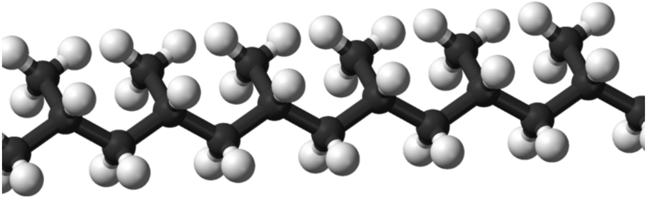

Figure: The structure of a polypropilen. *From: http://sl.wikipedia.org*

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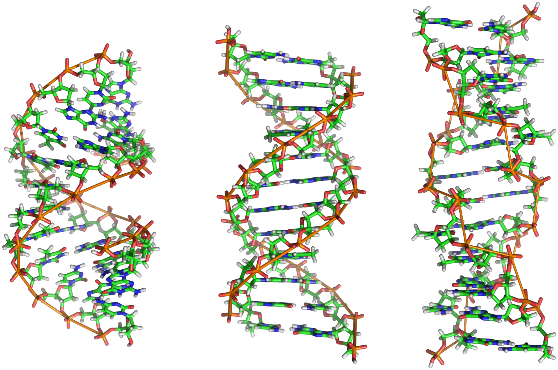


Figure: The structure of a DNA molecule depending on its environment.
From: http://en.wikipedia.org

Polygraphs

Definition. *A* *polygraph* $\Omega_n = \Omega_n(G_1, \ldots G_n; X_1, \ldots X_n)$ *over mutually disjoint monographs* $G_1, \ldots, G_n$ *has the vertex set*

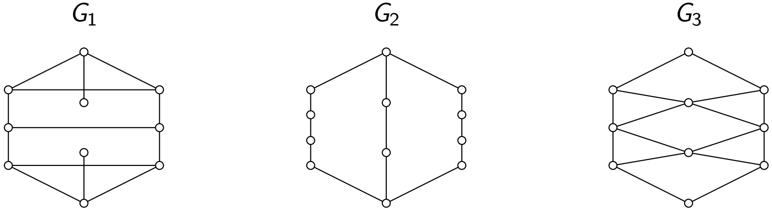
$$V(\Omega_n) = V(G_1) \cup \ldots \cup V(G_n),$$

and the edge set

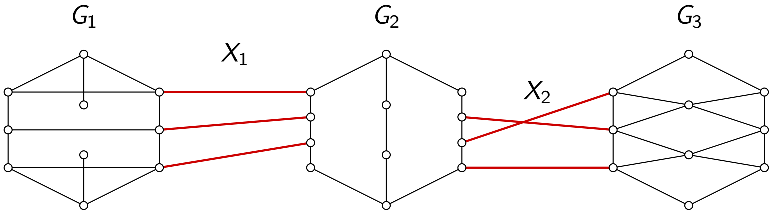
$$E(\Omega_n) = E(G_1) \cup X_1 \cup \ldots \cup E(G_n) \cup X_n,$$

where $X_i \subseteq V(G_i) \times V(G_{i+1})$ *for* $i = 1, \ldots, n$ *and* $G_{n+1} \cong G_1$*.*

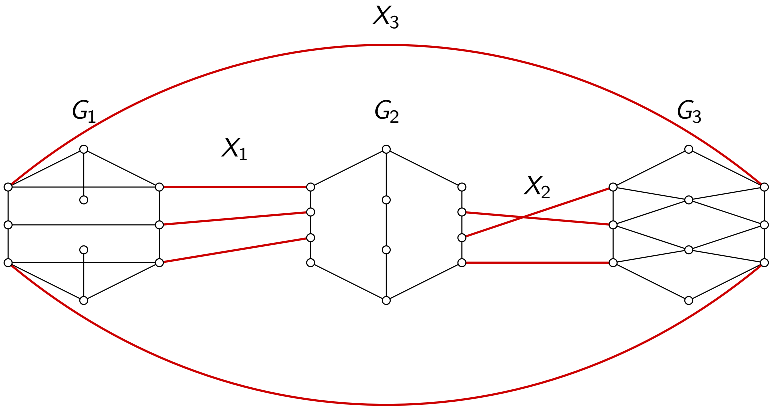
Polygraphs



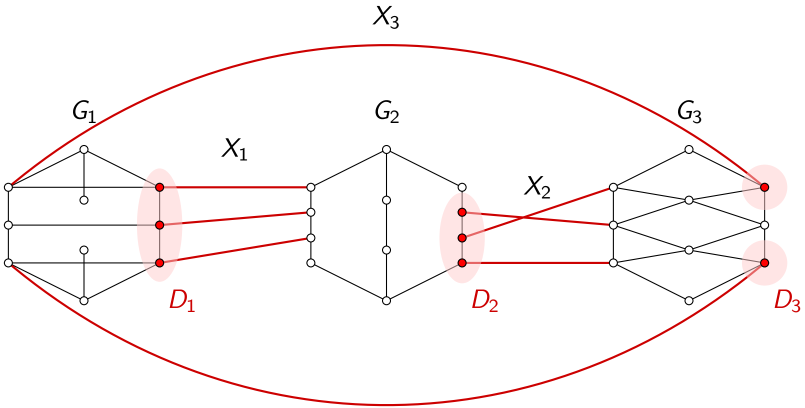
Polygraphs



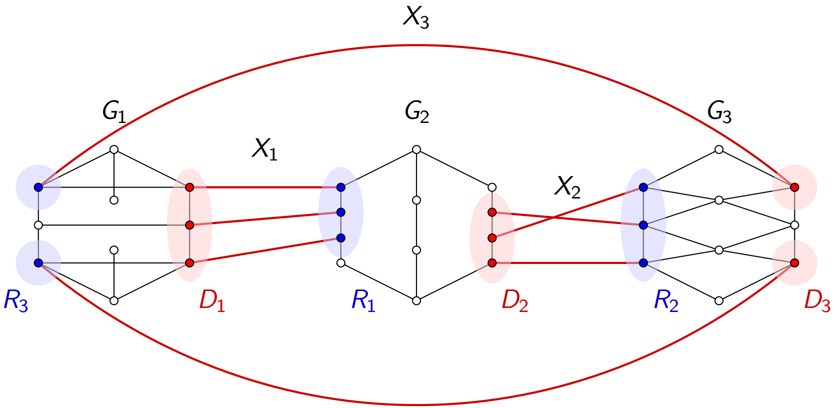
Polygraphs



Polygraphs

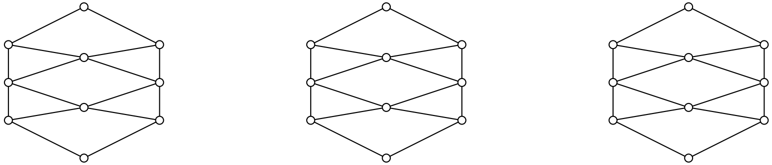


Polygraphs

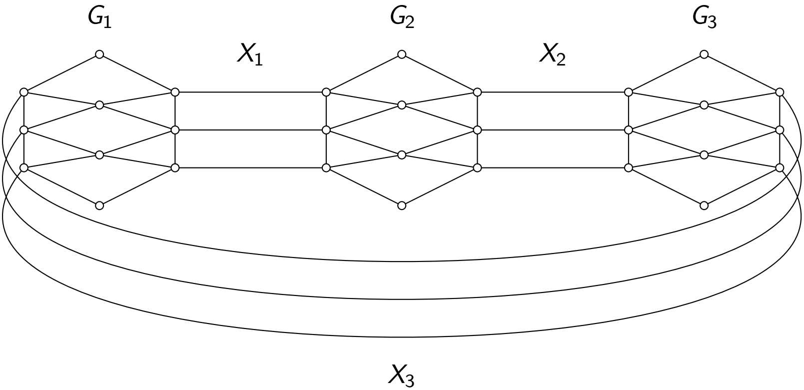


Rotagraph

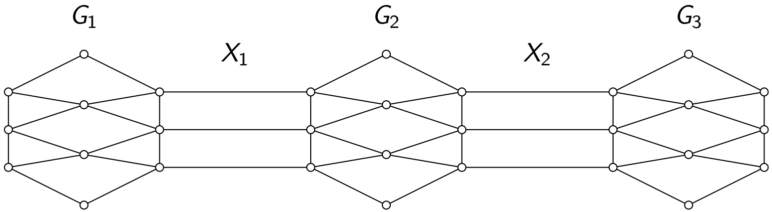
G_1 G_2 G_3



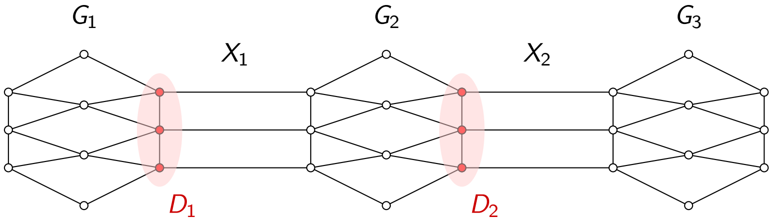
Rotagraph



Fasciagraph



Fasciagraph



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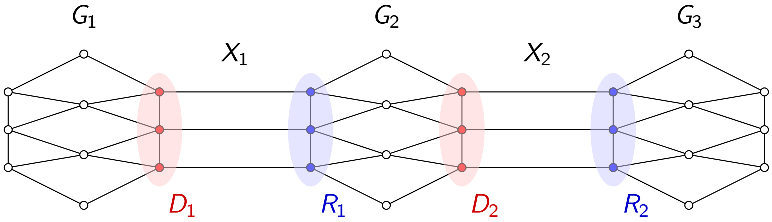
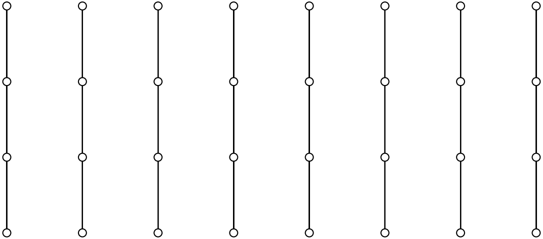


Graph products



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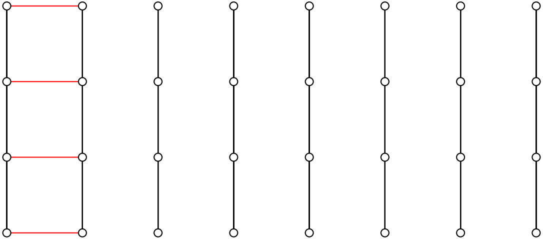
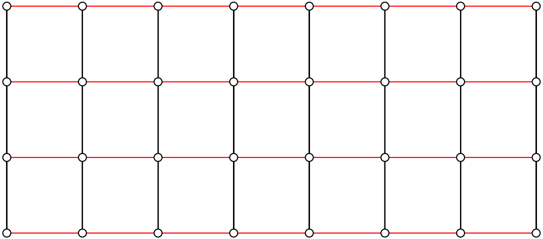


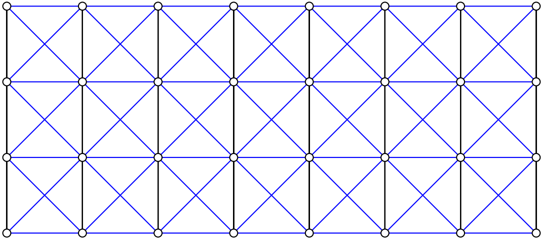
Graph products

$$P_n \square P_k$$



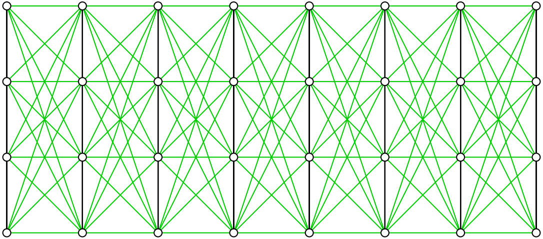
Graph products

$P_n \boxtimes P_k$



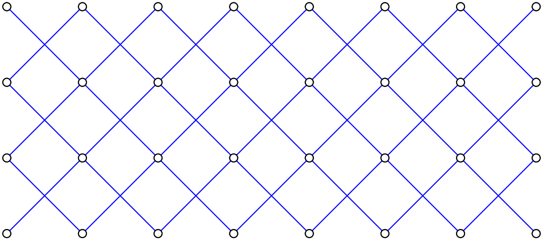
Graph products

$P_n [P_k]$



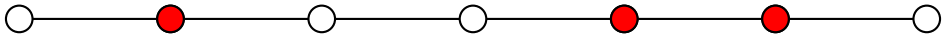
Graph products

$P_n \times P_k$



Domination number

Definition. *A set $D \subseteq V$ of a graph $G = (V, E)$ is a dominating set, if $N[D] = V$. The size of the smallest dominating set of a graph is the domination number, $\gamma(G)$.*



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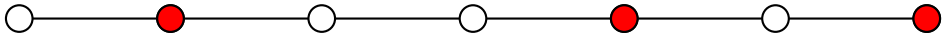
- independent domination number, $\gamma_i(G)$;



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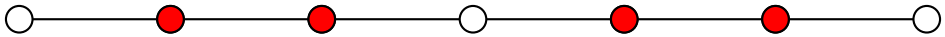
- independent domination number, $\gamma_i(G)$;
- total domination number, $\gamma_t(G)$;



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- independent domination number, $\gamma_i(G)$;
- total domination number, $\gamma_t(G)$;
- ...

Roman domination number

Definition. (Cockayne, Dreyer, S. M. and S.T. Hedetniemi, 2004)
A Roman dominating function (RDF) of a graph G is a function $f : V(G) \to \{0, 1, 2\}$ *such that every vertex u, for which* $f(u) = 0$*, is adjacent to at least one vertex v, for which* $f(v) = 2$*. The weight of a Roman dominating function is the value*

$$w(f) = \sum_{v \in V(G)} f(v).$$

The minimum weight of a Roman dominating function of a graph G is called Roman domination number, $\gamma_R(G)$.

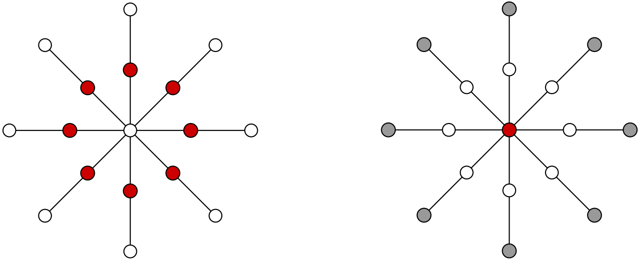

Figure: A γ-set and a γ_R-function of $\overline{S}_{1,8}$.

Algorithms for the domination problems

Theorem. (Johnson, 1979)
DOMINATING SET is NP-complete.

Proof. Reduction from 3-SAT.

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Still NP-complete for

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Still NP-complete for

- bipartite graphs (Chang et al., 1984), chordal graphs (Booth and Johnson, 1985),...
- other domination types.

Path algebra

A set P together with two binary operations $\oplus$ and $\circ$:

1. $(P, \oplus)$ is closed, associative and commutative with $e^{\oplus}$ as a unit;
2. $x \oplus x = x$ for all $x \in P$.
3. $(P, \circ)$ is closed and associative with e° as a unit;
4. $\circ$ is left– and right–distributive over $\oplus$;
5. for every $x \in P$, $x \circ e^{\oplus} = e^{\oplus} = e^{\oplus} \circ x$;

Path algebra

Examples of path algebras:

1. $\mathcal{P}_1 = (\{0,1\}, \max, \min, 0, 1)$; ⟵ Boolean algebra
2. $\mathcal{P}_2 = (\mathbb{N}_0 \cup \{-\infty\}, \max, +, -\infty, 0)$;
3. $\mathcal{P}_3 = (\mathbb{N}_0 \cup \{\infty\}, \min, +, \infty, 0)$. ⟵ Tropical semiring

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Matrix with elements of a path algebra

$\mathcal{M}_n(\mathcal{P}) \ldots n \times n$ matrices over P, where $\mathcal{P} = (P, \oplus, \circ, e^{\oplus}, e^{\circ})$.

$$
\begin{aligned}
(A \oplus_M B)_{ij} &= A_{ij} \oplus B_{ij}, \\
(A \circ_M B)_{ij} &= \bigoplus_{k=1}^{n} A_{ik} \circ B_{kj}.
\end{aligned}
$$

Observation: $\mathcal{M}_n(\mathcal{P})$ equipped with above operations is a path algebra with the zero and the unit matrix as units of semiring.

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Example: $\mathcal{P} = (\mathbb{N}_0 \cup \{\infty\}, \min, +, \infty, 0)$.

- $(A \oplus B)_{ij} = \min\{A_{ij}, B_{ij}\}$,
- $(A \circ B)_{ij} = \min_{k=1,\dots,n}\{A_{ik} + B_{kj}\}$
- The zero matrix:

$$\begin{bmatrix} \infty & \infty & \dots & \infty \\ \vdots & \vdots & \ddots & \vdots \\ \infty & \infty & \dots & \infty \end{bmatrix}$$

- The unit matrix:

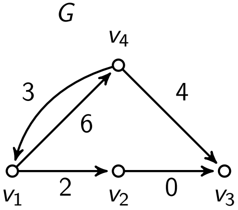
$$\begin{bmatrix} 0 & \infty & \dots & \infty \\ \infty & 0 & \dots & \infty \\ \vdots & \vdots & \ddots & \vdots \\ \infty & \infty & \dots & 0 \end{bmatrix}$$

Labeled digraphs

- A digraph G, $V(G) = \{v_1, v_2, \ldots, v_n\}$;
- A path algebra $\mathcal{P} = (P, \oplus, \circ, e^{\oplus}, e^{\circ})$;
- $\ell : E(G) \longrightarrow P$ a labelling.

$$(A(G))_{ij} = \begin{cases} \ell(v_i, v_j); & \text{if } (v_i, v_j) \text{ is an arc of } G \\ e^{\oplus}; & \text{otherwise} \end{cases}$$

Labeled digraphs



	v_1	v_2	v_3	v_4
v_1	∞	2	∞	6
v_2	∞	∞	0	∞
v_3	∞	∞	∞	∞
v_4	3	∞	4	∞

Labelling on walks

For a walk $Q = (v_{i_0}, v_{i_1})(v_{i_1}, v_{i_2})\ldots(v_{i_{k-1}}, v_{i_k})$ of G let

$$\ell(Q) = \ell\left(v_{i_0}, v_{i_1}\right) \circ \ell\left(v_{i_1}, v_{i_2}\right) \circ \ldots \circ \ell\left(v_{i_{k-1}}, v_{i_k}\right).$$

Observation: Let S_{ij}^k be the set of all walks of order k from v_i to v_j in G. Then

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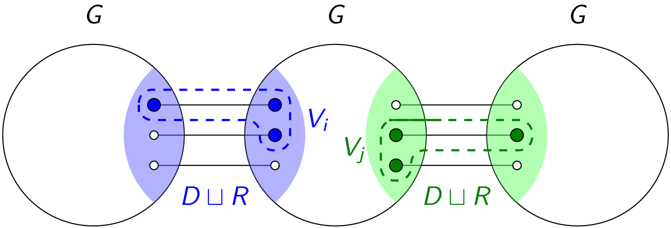
The algorithm

Different graph invariants (Klavžar and Žerovnik, 1996)

Let $\omega_n(G;X)$ be a rotagraph and $\psi_n(G;X)$ a fasciagraph. Define a labeled digraph $\mathcal{G} = \mathcal{G}(G;X)$:

- $V(\mathcal{G}) \ldots$ subsets of $D \sqcup R$;
- $E(\mathcal{G}) \ldots$ between vertices that are not in "conflict".

The algorithm



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2. Form $A(\mathcal{G})$ and in $\mathcal{M}(\mathcal{P})$ calculate $A(\mathcal{G})^n$. $\longleftarrow O(\log n)$
3. Among admissable coefficients of $A(\mathcal{G})^n$ select one that optimizes the corresponding goal function.

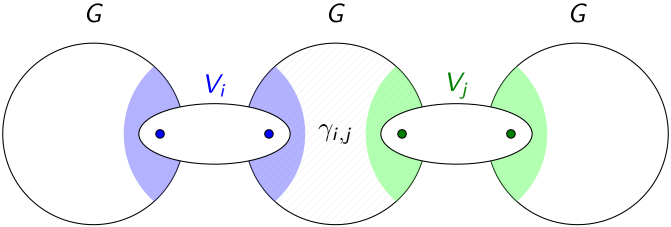
The domination number

Define a labeled digraph $\mathcal{G} = \mathcal{G}(G; X)$:

- $V(\mathcal{G}) \ldots$ subsets of $R \sqcup D$;
- $E(\mathcal{G}) \ldots$ all possible;
- Path algebra: a tropical semiring $(\mathbb{N}_0 \cup \{\infty\}, \min, +, \infty, 0)$.

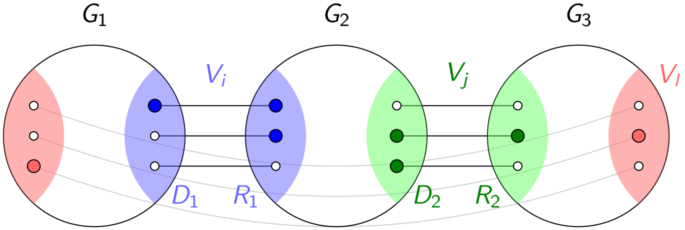
The domination number

$$\ell(V_i, V_j) = |V_i \cap R| + \gamma_{i,j}(G; X) + |D \cap V_j| - |V_i \cap R \cap D \cap V_j|$$



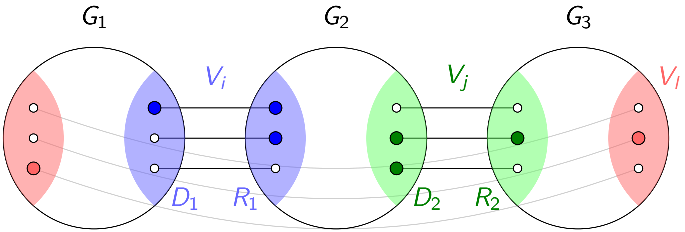
The domination number

$$\left(A(G)^3\right)_{ll} = \min_{i,j}\{\ell(V_l, V_i) + \ell(V_i, V_j) + \ell(V_j, V_l)\}$$



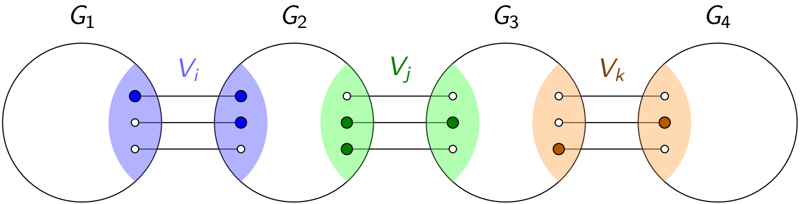
The domination number

$$\gamma\left(\omega_n(G;X)\right) = \min_{I}\left(A\left(\mathcal{G}\right)^n\right)_{II}.$$



The domination number

$$\gamma\left(\psi_n(G;X)\right) = \left(A\left(\mathcal{G}\right)^n\right)_{00}$$



Results

- Other domination - type invariants;
- Time complexity;
- Space complexity;
- Implementation to get some closed expressions.

The Roman domination number

- $V(\mathcal{G}) = \{(V_i, W_i) \mid V_i, W_i \subseteq U, V_i \cap W_i = \emptyset\}$;
- $((V_i, W_i), (V_j, W_j)) \in E(\mathcal{G})$ iff

$$R \cap V_i \cap W_j \cap D = \emptyset$$

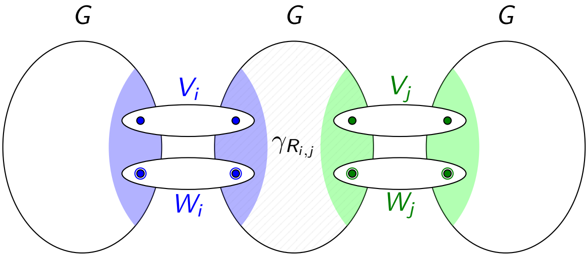
  and

$$R \cap W_i \cap V_j \cap D = \emptyset;$$

- Path algebra: a tropical semiring $(\mathbb{N}_0 \cup \{\infty\}, \min, +, \infty, 0)$.

The Roman domination number

$$\ell(v_i, v_j) = |R \cap V_i| + 2\,|R \cap W_i| + |V_j \cap D| + 2\,|W_j \cap D| -$$
$$- |R \cap V_i \cap V_j \cap D| - 2\,|R \cap W_i \cap W_j \cap D| + \gamma_{R_{i,j}}(G; X).$$



The Roman domination number

Then

$$\gamma_R\left(\psi_n(G;X)\right) = \left(A\left(\mathcal{G}\right)^n\right)_{00}$$

and

$$\gamma_R\left(\omega_n(G;X)\right) = \min_i \left(A\left(\mathcal{G}\right)^n\right)_{ii}.$$

Algorithm The Roman domination number

1. For a path algebra select $\mathcal{P} = (\mathbb{N}_0 \cup \{\infty\}, \min, +, \infty, 0)$.
2. Label $\mathcal{G} = \mathcal{G}(G; X)$ as defined above.
3. In $\mathcal{M}(\mathcal{P})$ calculate $A(\mathcal{G})^n$.
4. Let $\gamma_R(\psi_n(G;X)) = \left(A(\mathcal{G})^n\right)_{00}$ and $\gamma_R(\omega_n(G;X)) = \min_i \left(A(\mathcal{G})^n\right)_{ii}$.

Theorem. (-, Žerovnik, 2012) The Algorithm correctly computes the Roman domination number of rotagraphs and fasciagraphs:

$$\gamma_R\left(\psi_n(G;X)\right) = \left(A\left(\mathcal{G}\right)^n\right)_{00}$$

$$\gamma_R\left(\omega_n(G;X)\right) = \min_i \left(A\left(\mathcal{G}\right)^n\right)_{ii}$$

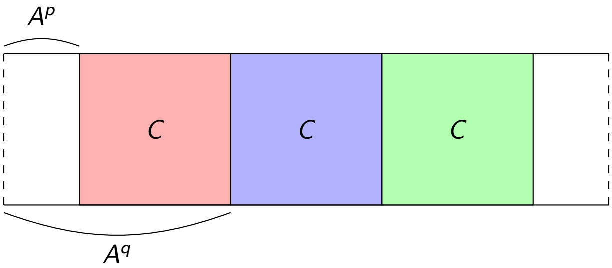
in $O(\log n)$ time.

Time complexity

Theorem. (-, Žerovnik, 2012) Let $A \in \mathcal{M}_n(\mathcal{P})$, where $\mathcal{P} = (\mathbb{N}_0 \cup \{\infty\}, \min, +, \infty, 0)$ is a path algebra. Let $N = |V(\mathcal{G}(G;X))|$, $K = |V(G)|$ and $a = \max\{a_0, \ldots, a_l\}$. Then there is an index $q \leq (2aK+2)^{N^2}$ such that $A^q = A^p + C$ for some index $p < q$ and some constant matrix $C = [c]_{ij}$. Let $P = q - p$. Then for every $r \geq p$ and every $s \geq 0$ we have

$$A^{r+sP} = A^r + sC\,.$$

Time complexity



Space complexity

$$\gamma_R\left(\psi_n(G;X)\right) = \left(A\left(\mathcal{G}\right)^n\right)_{00}$$

$$A^n_{0i} = \min_k \left\{A^{n-1}_{0k} + A_{ki}\right\}.$$

Lemma. (-, Žerovnik, 2012) Assume that the j–th row of A^{n+P} and A^n differ for a constant, $a^{n+P}_{ji} = a^n_{ji} + C$ for all i. Then $\min_i a^{n+P}_{ji} = \min_i a^n_{ji} + C$.

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Space complexity

Lemma. (-, Žerovnik, 2012) Let $A^q = A^p + C$ and $P = q - p$. Then for every $t \in 0, 1, \ldots, P-1$ there is a constant C_t such that for all $n \geq p$ with $t \equiv (n-p) \pmod{P}$ we have

$$\gamma_R(\psi_n(G;X)) - \gamma_R(\omega_n(G;X)) = C_t.$$

Conclusion

Corollary. (-, Žerovnik, 2012) The domination number and its varieties of fasciagraphs and rotagraphs can be computed in constant time, i.e. independently of the size of a monograph G.

k	$\gamma(P_n \square C_k)$	
3	$\left\lceil \frac{3n}{4} \right\rceil + 1$;	if $n \equiv 0 \pmod 4$
	$\left\lceil \frac{3n}{4} \right\rceil$;	otherwise
4	n	
5	3;	if $n = 2$
	4;	if $n = 3$
	$n + 2$;	otherwise
6	$\left\lceil \frac{4n}{3} \right\rceil$;	if $n \equiv 1 \pmod 3$
	$\left\lceil \frac{4n}{3} \right\rceil + 1$;	otherwise
7	$\left\lceil \frac{3n}{2} \right\rceil + 1$;	if $n \equiv 1 \pmod 2$
	$\left\lceil \frac{3n}{2} \right\rceil + 2$;	otherwise

k	$\gamma(P_n \square C_k)$
8	$\begin{cases} 4; & \text{if } n = 2 \\ 6; & \text{if } n = 3 \\ 8; & \text{if } n = 4 \\ \left\lfloor \frac{9n}{5} \right\rfloor + 1; & \text{if } n \equiv 5 \pmod{10} \\ \left\lfloor \frac{9n}{5} \right\rfloor + 2; & \text{otherwise} \end{cases}$
9	$\begin{cases} 2n + 1; & \text{if } n = 2, 3 \\ 2n + 2; & \text{otherwise} \end{cases}$
10	$\begin{cases} 2n + 2; & \text{if } n \leq 5 \\ 2n + 3; & \text{if } 6 \leq n \leq 9 \\ 2n + 4 & \text{otherwise} \end{cases}$
11	$\begin{cases} \left\lceil \frac{19n}{8} \right\rceil + 1; & \text{if } n \in \{1, 2, 4, 6\} \text{ or } n \equiv 3 \pmod{8} \\ \left\lceil \frac{19n}{8} \right\rceil + 2; & \text{otherwise} \end{cases}$

k	$\gamma(C_n \square P_k)$	$\gamma(C_n \square C_k)$
3	$\left\lceil \frac{3n}{4} \right\rceil$	$\left\lceil \frac{3n}{4} \right\rceil$
4	$n+1$; if $n \in \{5, 9\}$ n; otherwise	n
5	4; if $n = 3$ $\left\lceil \frac{6n}{5} \right\rceil + 1$; if $n \equiv 3, 5, 9 \pmod{10}$ $\left\lceil \frac{6n}{5} \right\rceil$; otherwise	n; if $n \equiv 0 \pmod 5$ $n+2$; if $n \equiv 3 \pmod 5$ $n+1$; otherwise
6	9; if $n = 6$ $\left\lceil \frac{10n}{7} \right\rceil + 1$; $n \equiv 2, 6, 7, 9$ $13 \pmod{14}$ $\left\lceil \frac{10n}{7} \right\rceil$; otherwise	$\left\lceil \frac{4n}{3} \right\rceil + 1$; $n \equiv 2, 3, 8, 9 \pmod{18}$ $11, 14, 15, 17 \pmod{18}$ $\left\lceil \frac{4n}{3} \right\rceil$; otherwise
7	6; if $n = 3$ 16; if $n = 9$ 36; if $n = 21$ $\left\lceil \frac{5n}{3} \right\rceil$; otherwise	$\left\lceil \frac{3n}{2} \right\rceil$; $n \equiv 0, 5, 9 \pmod{14}$ $\left\lceil \frac{3n}{2} \right\rceil + 2$; $n \equiv 2, 8, 12 \pmod{14}$ $\left\lceil \frac{3n}{2} \right\rceil + 1$; otherwise

k	$\gamma_R(P_n \Box P_k)$	$\gamma_R(P_n \Box C_k)$
2	$n+1$	
3	$\left\lceil \frac{3n}{2} \right\rceil + 2$; if $n \equiv 3 \pmod 4$ $\left\lceil \frac{3n}{2} \right\rceil + 1$; otherwise	$\left\lfloor \frac{3n}{2} \right\rfloor + 1$
4	$2n+1$; if $n \in \{1,2,3,5,6\}$ $2n$; otherwise	$2n$
5	8; if $n = 3$ $\left\lfloor \frac{12n}{5} \right\rfloor + 2$; otherwise	$2n+2$
6	$\left\lceil \frac{14n}{5} \right\rceil + 2$; if $n < 5$ or $n \equiv 0,3,4 \pmod 5$ $\left\lceil \frac{14n}{5} \right\rceil + 3$; otherwise	$\left\lfloor \frac{8n}{3} \right\rfloor + 2$

k	$\gamma_R(P_n \Box P_k)$	$\gamma_R(P_n \Box C_k)$
7	$\left\lfloor \frac{16n}{5} \right\rfloor + 2$; if $n \in \{1, 2, 4, 7\}$ or $n \equiv 0 \pmod 5$ $\left\lfloor \frac{16n}{5} \right\rfloor + 3$; otherwise	$3n + 2$; if $n \in \{1, 2, 4\}$ $3n + 3$; otherwise
8	9; if $n = 2$ 16; if $n = 4$ $\left\lfloor \frac{18n}{5} \right\rfloor + 4$; if $n \equiv 3 \pmod 5$ $\left\lfloor \frac{18n}{5} \right\rfloor + 3$; otherwise	8; if $n = 2$ $\left\lfloor \frac{7n}{2} \right\rfloor + 2$; if $n \in \{3, 4, 8\}$ $\left\lfloor \frac{7n}{2} \right\rfloor + 3$; otherwise

k	$\gamma_R(C_n \Box P_k)$		$\gamma_R(C_n \Box C_k)$	
3	5;	$n = 3$	$\left\lceil \frac{3n}{2} \right\rceil$	
	$\left\lceil \frac{3n}{2} \right\rceil$;	if $n \equiv 0, 1 \pmod 4$		
	$\left\lceil \frac{3n}{2} \right\rceil + 1$;	otherwise		
4	7;	if $n = 3$	$2n$	
	$2n$;	otherwise		
5	$\left\lceil \frac{12n}{5} \right\rceil + 1$;	if $n \equiv 2 \pmod 5$	$2n$;	if $n \equiv 0 \pmod 5$
	$\left\lceil \frac{12n}{5} \right\rceil$;	otherwise	$2n + 2$;	otherwise
6	$\left\lfloor \frac{14n}{5} \right\rfloor$;	if $n \equiv 0 \pmod 5$	$\left\lfloor \frac{8n}{3} \right\rfloor$;	if $n \equiv 3, 5, 8, 11, 13, 17 \pmod{18}$
	$\left\lfloor \frac{14n}{5} \right\rfloor + 1$;	if $n \equiv 4 \pmod 5$	$\left\lfloor \frac{8n}{3} \right\rfloor + 1$;	if $n \equiv 4 \pmod 5$
	$\left\lfloor \frac{14n}{5} \right\rfloor + 2$;	otherwise	$\left\lfloor \frac{8n}{3} \right\rfloor + 2$;	otherwise

Thank you for your attention!