Solvable regular covering projections of graphs

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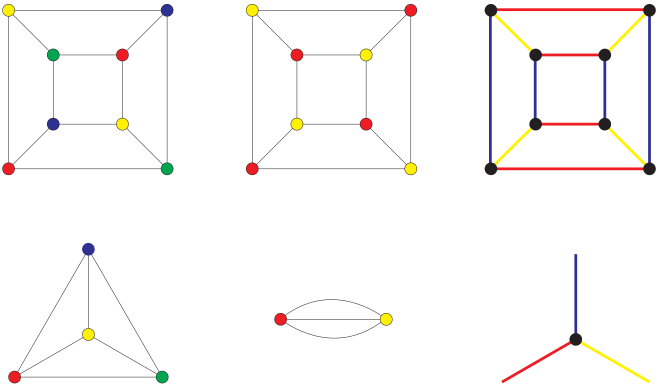
Mathematical Research Seminar
UP FAMNIT

November 3, 2014

Graph covers

Regular coverings of connected graphs

a surjective mapping $p\colon \tilde{X} \to X$ s.t.
fibres $p^{-1}(v)$ = orbits of a semiregular subgroup $\mathrm{CT}_p \leq \mathrm{Aut}(\tilde{X})$



Distinguishing covers one from another

Isomorphism of regular coverings

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In particular, $g = \mathrm{id} \Rightarrow p$ and p' are **equivalent**.

Covers, combinatorially

The derived graph

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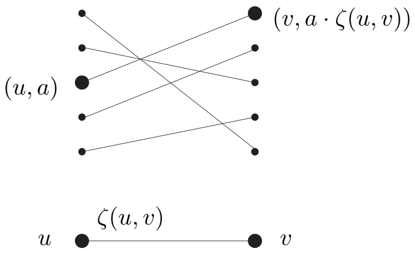
graph $X \times_\zeta \Gamma$:
vertex set $V(X) \times \Gamma$,
$(u,a) \sim (v, a \cdot \zeta(u,v))$ for $u \sim v$ in X



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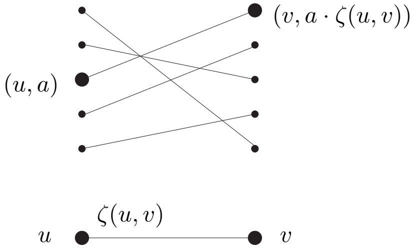

The projection $p_\zeta\colon X \times_\zeta \Gamma \to X$ onto the first coordinate

regular covering projection

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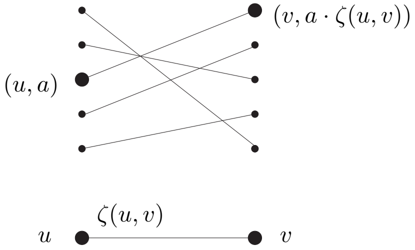

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regular covering projection

Regular covering projection $p\colon \tilde{X} \to X$

reconstructed by voltages $\Gamma \cong \mathrm{CT}_p$

Symmetries of covering graph vs. base graph

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Lifting automorphisms along regular covering projections

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all elements of $G \leq \mathrm{Aut}(X)$ lift

G-admissible regular cover

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Applications

classification of particular classes of graphs and maps on surfaces,
counting the number of graphs in certain families,
constructing infinite families or produce catalogues of graphs
with prescribed degree of symmetry up to a certain reasonable size

The structure of the lifted group

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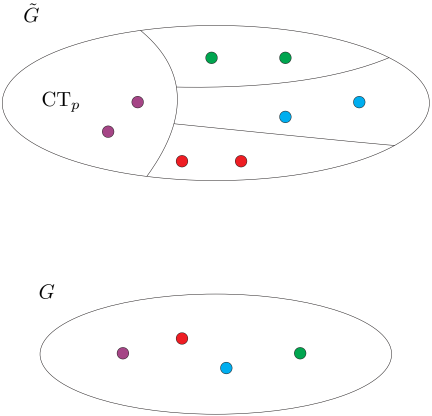

$\tilde{G}$ is a group extension of CT_p by G
$\mathrm{CT}_p \lhd \tilde{G}$ and $\tilde{G}/\mathrm{CT}_p \cong G$

Universal covering projection

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covering projection $p^* : \mathcal{T} \to X$

where $\mathcal{T}$ is a tree

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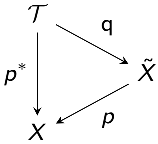
Universal property

for every $p\colon \tilde{X} \to X$ there exists a unique $q\colon \mathcal{T} \to \tilde{X}$ s.t.



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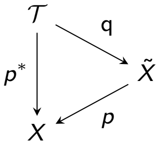


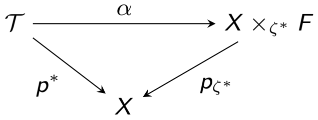
for each $G \leq \mathrm{Aut}(X)$

p^* is G-admissible

Universal covering projection, combinatorially

Reconstruction of $p^*\colon \mathcal{T} \to X$

choose a spanning tree T_X in X rooted at u_0,
let $\{x_1, \ldots, x_r\} \subset A(X)$ contain exactly one arc from each edge of $E(X \setminus T_X)$,
take $F = \langle a_1, \ldots, a_r \rangle \cong \pi(X, u_0)$ as a voltage group,
define $\zeta^*\colon A(X) \to F$ to be trivial on $A(T_X)$ and $\zeta^*(x_i^{\pm}) = a_i^{\pm}$,



identify $\mathrm{CT}_{p_{\zeta^*}}$ with F via $\tilde{\mathrm{id}}_a(u, c) = (u, ac)$

Normal subgroups of $F \longleftrightarrow$ regular covering projections

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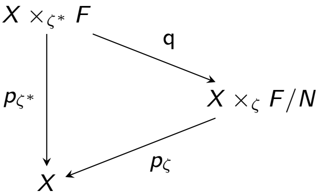
Take $N \triangleleft F$



$$\zeta\colon X \to F/N,\ \zeta(u,v) = \zeta^*(u,v)N$$

$$\mathrm{CT}_{p_\zeta} \cong F/N$$

$$\mathrm{CT}_q \cong N \cong \pi(X \times_{\zeta^*} F/N, (u_0, 1))$$

Which normal subgroups in F give rise to G-admissibility?

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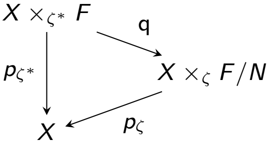

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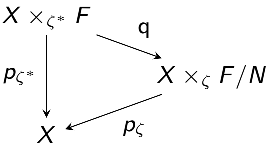

$$G^* = \text{the lifted group of } G \text{ along } p_{\zeta^*} (F \lhd G^* \text{ and } G^*/F \cong G)$$
$$N \lhd F \lhd G^*$$

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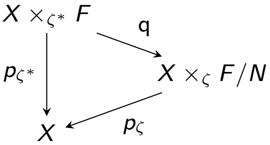

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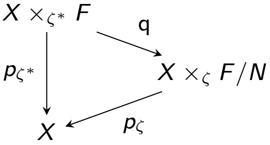

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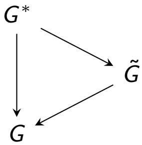
Then q is $\tilde{G}$ – *admissible*



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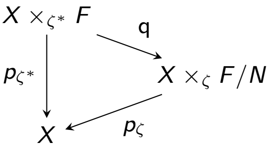

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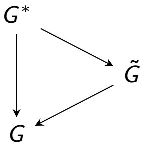

$N \lhd G^*$ **s.t.** $N \leq F \longleftrightarrow$ G**-admissible regular coverings**

Finding a presentation of G^*

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$$\textbf{Suppose } G = \langle g_1, \ldots, g_n \mid r_1(g_1, \ldots, g_n), \ldots, r_m(g_1, \ldots, g_n) \rangle$$

for each g_i choose the unique lift $\overline{g_i}$ with $\overline{g_i}(u_0, 1) = (g_i u_0, 1)$,

$$\overline{g_i}(v, a_{i_1}^{\pm} \cdots a_{i_l}^{\pm}) = (g_i v, (\zeta^* g_i W^{i_1})^{\pm} \cdots (\zeta^* g_i W^{i_l})^{\pm} (\zeta^* g_i Q)^{-1}),$$

where W^{i_j} is the fundamental u_0-based closed walk determined by x_{i_j} and T_X,
$Q\colon v \to u_0$ the unique path in T_X,

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Since F **is normal in** G^*

let $\overline{g_i} a_j \overline{g_i}^{-1} = w_{i,j}(a_1, \ldots, a_r) \in F$,

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Up to a prescribed order n of the respective covering graphs

find all $N \lhd G^*$ contained in F with F/N solvable of order at most n

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find all $N \triangleleft G^*$ contained in F with F/N solvable of order at most n

The basic idea

in a solvable F/N there exists a normal elementary abelian subgroup K/N;
if K is known, N can be found by considering $H \triangleleft G^*$
with $H \leq K$ and K/H elementary abelian

An algorithm

An algorithm

Computing normal subgroups with solvable factor

Input: a finitely presented group G, a normal subgroup H of G given by words in the generators of G that generates H, an integer $n > 0$

Output: the set $\mathcal{N}$ of all normal subgroups N of G contained in H with H/N solvable of order at most n

Set $\mathcal{N} = \{H\}$ and set $\mathit{Processed} = \emptyset$;
while $\mathcal{N} \setminus \mathit{Processed} \neq \emptyset$ **do**
  Choose $K \in \mathcal{N} \setminus \mathit{Processed}$ and insert K in $\mathit{Processed}$;
  foreach prime p with $p|H : K| \leq n$ **do**
    Let $M = K/[K, K]\, K^p$ with $f : K \to M$ the natural epimorphisms;
    Turn M into $\mathbb{Z}_p[G/K]$-module;
    Find the set $\mathcal{S}$ of all maximal $\mathbb{Z}_p[G/N]$-submodules of M whose codimension d satisfies $p^d|H : K| \leq n$;
    foreach $S \in \mathcal{S}$ **do**
      Let $L = f^{-1}(S)$;
      if L is not equal to any of subgroups in $\mathcal{N}$ **then**
        Insert L into $\mathcal{N}$;
return $\mathcal{N}$;

Thank you!