LINEAR CRYPTANALYSIS

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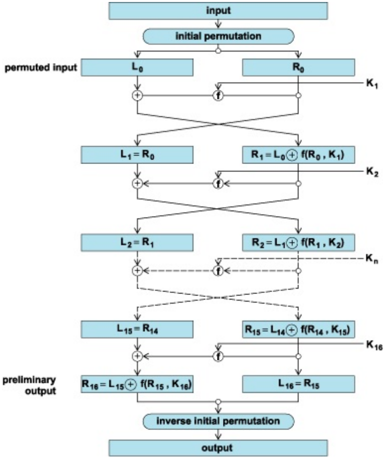
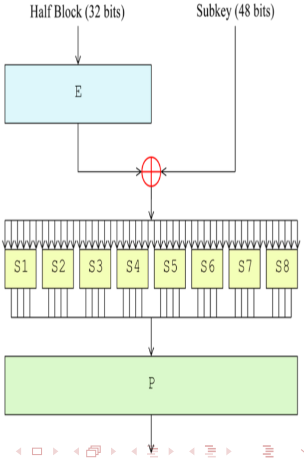
Data Encryption Standard (DES)-history

- The most famous and analyzed block cipher - international banking standard
- Design criteria were kept secret (including differential cryptanalysis) for more than 20 years
- Based on IBM's LUCIFER but changes introduced by NSA
- Still no trapdoors have been found

DES Algorithm

Block length 64 bits, key length 56 bits





Linear Attack on DES idea

In DES S-box maps 6 input bits to 4 output bits i.e.,

$$S : F_2^6 \to F_2^4, \text{ where } F_2 = \{0, 1\}$$

$$(x_1, ..., x_6) \to (y_1, ..., y_4)$$

Assume,

$$y_1 = x_1x_2x_3 + x_2x_5 + x_2 + x_4$$

$$y_2 = x_1x_2x_3 + x_2x_5 + x_1 + x_4$$

$$\Rightarrow y_1 \oplus y_2 = x_1 \oplus x_2$$

Specifically, if P_i are plaintext bits, C_i are ciphertext bits, and K_i are subkey bits, then we wish to find an expression of the form

$$P_{i_1} \oplus P_{i_2} \oplus ...P_{i_j} \oplus C_{i_1} \oplus C_{i_2} \oplus ...C_{i_k} = K_{i_1} \oplus K_{i_2} \oplus ...K_{i_m}$$

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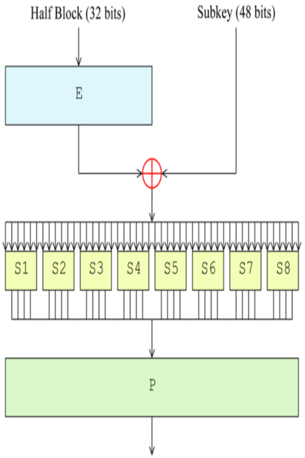
Definition

For a given linear approximation to part of a cipher, let p be the probability that it holds. We refer to $|p - \frac{1}{2}|$ as the **bias** of the approximation.

Reducing DES



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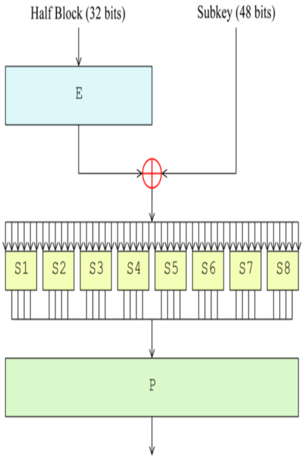
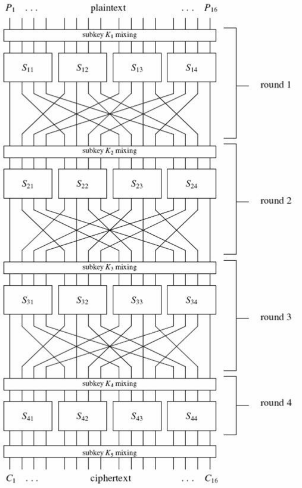


Let us simplify DES by

- Reducing the code length from 64 to 16
- Removing expansion function $E(R)$
- Taking identical S-boxes of size 4 bits
- Repeating over less rounds

Substitution - Permutation Network



- SPN is a 16-bit block cipher
- Each round consists of a substitution and a permutation
- 4×4-bit S-box
- key-mixing by XOR-ing

SPN S-box and Permutation

SPN uses a single 4-bit S-Box that has the following structure:

input	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
output	E	4	D	1	2	F	B	8	3	A	6	C	5	9	0	7

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And the following 16-bit permutation:

input	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
output	1	5	9	13	2	6	10	14	3	7	11	15	4	8	12	16

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The S-Box provides the *confusion* function and the permutation implements the *diffusion* operation in SPN, thus making it cryptographically similar to DES.

Linear and Affine approximation of S-Box

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To find the linear or affine approximation of the S-Box we simply consider every possible expression of the input bits X_i and output bits Y_j . Thus the expression has the form

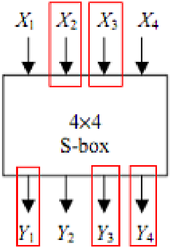
$$\bigoplus_{i \in U} X_i = \bigoplus_{j \in V} Y_j$$

where U and V range over all possible subsets of $\{1, 2, 3, 4\}$. We then compare how often this expression coincides with the S-Box.

The number of agreements (minus 8) between the S-Box and every possible expression is summarized in the table below. Thus to get the bias, one must only divide by 16.

Input Sum \ Output Sum	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
0	+8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	−2	−2	0	0	−2	+6	+2	+2	0	0	+2	+2	0	0
2	0	0	−2	−2	0	0	−2	−2	0	0	+2	+2	0	0	−6	+2
3	0	0	0	0	0	0	0	0	+2	−6	−2	−2	+2	+2	−2	−2
4	0	+2	0	−2	−2	−4	−2	0	0	−2	0	+2	+2	−4	+2	0
5	0	−2	−2	0	−2	0	+4	+2	−2	0	−4	+2	0	−2	−2	0
6	0	+2	−2	+4	+2	0	0	+2	0	−2	+2	+4	−2	0	0	−2
7	0	−2	0	+2	+2	−4	+2	0	−2	0	+2	0	+4	+2	0	+2
8	0	0	0	0	0	0	0	0	−2	+2	+2	−2	+2	−2	−2	−6
9	0	0	−2	−2	0	0	−2	−2	−4	0	−2	+2	0	+4	+2	−2
A	0	+4	−2	+2	−4	0	+2	−2	+2	+2	0	0	+2	+2	0	0
B	0	+4	0	−4	+4	0	+4	0	0	0	0	0	0	0	0	0
C	0	−2	+4	−2	−2	0	+2	0	+2	0	+2	+4	0	+2	0	−2
D	0	+2	+2	0	−2	+4	0	+2	−4	−2	+2	0	+2	0	0	+2
E	0	+2	+2	0	−2	−4	0	+2	−2	0	0	−2	−4	+2	−2	0
F	0	−2	−4	−2	−2	0	+2	0	0	−2	+4	−2	−2	0	+2	0

S-Box Approximation Example



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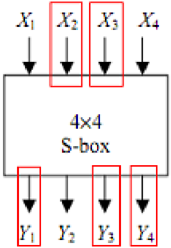

We can take the following expression as an example:

$$X_2 \oplus X_3 = Y_1 \oplus Y_3 \oplus Y_4$$

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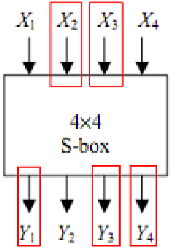

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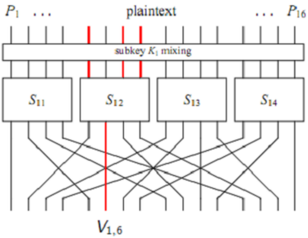
$$X_2 \oplus X_3 = Y_1 \oplus Y_3 \oplus Y_4$$

Applying all possible values for the input X bits it turns out that the expression holds in 12 out of the 16 cases. Hence, this expression has a bias of $\frac{12}{16} - \frac{1}{2} = \frac{1}{4}$

Linear Attack on SPN

What this means for 1 round



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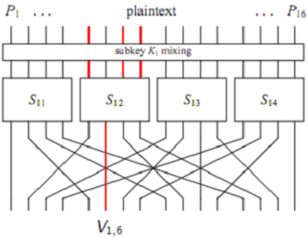

Note, from the previous table, that the expression $X_1 \oplus X_3 \oplus X_4 = Y_2$ has a bias of $+\frac{1}{4}$

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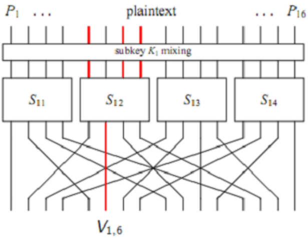

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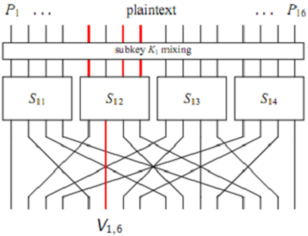
Note also that $U_5^1 = P_5 \oplus K_5^1$

Linear Attack on SPN

We can now write down the following linear approximation across the 1st round of SPN:



$$V_6^1 = U_5^1 \oplus U_7^1 \oplus U_8^1$$

$$= (P_5 \oplus K_5^1) \oplus (P_7 \oplus K_7^1) \oplus (P_8 \oplus K_8^1)$$

This expression holds with probability of $\frac{3}{4}$ (bias of $+\frac{1}{4}$)

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Piling - Up Lemma

For n independent, random binary variables $X_1, X_2, ..., X_n$,

$$Pr(X_1 \oplus X_2 \oplus ... \oplus X_n = 0) = \frac{1}{2} + 2^{n-1} \prod_{i=1}^{n} \varepsilon_i$$

or, equivalently,

$$\varepsilon_{1,2,...,n} = 2^{n-1} \prod_{i=1}^{n} \varepsilon_i$$

where $\varepsilon_{1,2,...,n}$ represents the bias of $X_1 \oplus X_2 \oplus ... \oplus X_n = 0$

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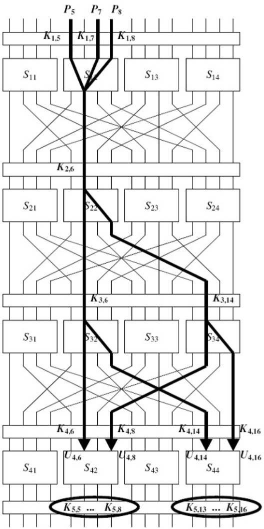

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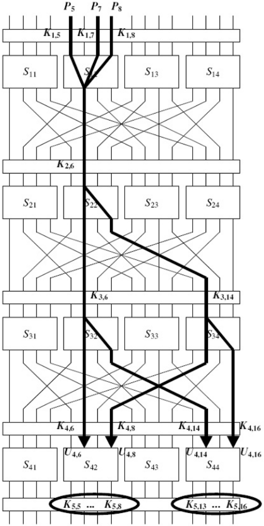

We can write down 4 approximations

$$S_{12} : X_1 \oplus X_3 \oplus X_4 = Y_2$$

$$S_{22} : X_2 = Y_2 \oplus Y_4$$

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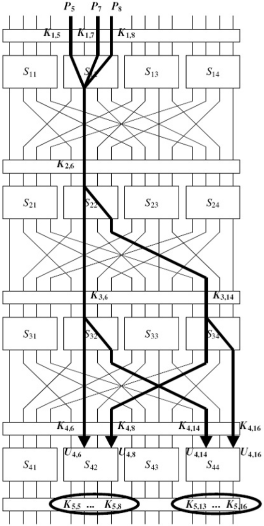

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Each of these has a probability bias magnitude of $\frac{1}{4}$.

Linear Attack on SPN

Consider the first 2 rounds:

$$X_1 \oplus X_3 \oplus X_4 = Y_2$$

$$\Rightarrow (P_5 \oplus K_5^1) \oplus (P_7 \oplus K_7^1) \oplus (P_8 \oplus K_8^1) = V_6^1$$

$$X_2 = Y_2 \oplus Y_4 \Rightarrow (V_6^1 \oplus K_6^2 = V_6^2 \oplus V_8^2)$$

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$$X_2 = Y_2 \oplus Y_4 \Rightarrow (V_6^1 \oplus K_6^2 = V_6^2 \oplus V_8^2)$$

Each of these has a bias of magnitude $\frac{1}{4}$ and we can combine to obtain:

$$V_6^2 \oplus V_8^2 \oplus P_5 \oplus P_7 \oplus P_8 \oplus K_5^1 \oplus K_7^1 \oplus K_8^1 \oplus K_6^2 = 0$$

By the Piling Up Lemma this holds with bias

$$2 \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{8}$$

Linear Attack on SPN

Using this principle we can write the following equation over 3 rounds of SPN:

$$U_6^4 \oplus U_8^4 \oplus U_{14}^4 \oplus U_{16}^4 \oplus P_5 \oplus P_7 \oplus P_8 = \overline{K}$$

Where
$\overline{K} = K_5^1 \oplus K_7^1 \oplus K_8^1 \oplus K_6^2 \oplus K_6^3 \oplus K_{14}^3 \oplus K_6^4 \oplus K_8^4 \oplus K_{14}^4 \oplus K_{16}^4$.

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The magnitude of the bias of the above expression, by the Piling Up Lemma, is $\frac{1}{32}$.

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For each value of the guessed target partial subkey we can undo the last round and determine the bias of the equation. Highest bias indicates likely correct guess.

Conclusion

- One might say that 8 bits of the last round key is not very useful
- How many plaintext/ciphertext pairs do we need to make this attack?

Thank you for your Attention!