High order parametric polynomial approximation of conic sections

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Raziskovalni matematični seminar

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Problem:

- Conic sections are standard objects in CAGD.
- A general conic is given with the equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

- Eigenvalues of the matrix

$$\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$$

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- Conics can be presented also by quadratic rational curves.
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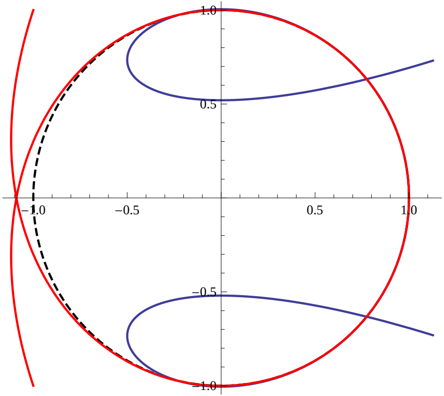
- We need good polynomial approximant.
- We will require the interpolation of at least one point on a conic and tangent direction at this point.
- But: such an approximant is “good” only close to the interpolation point.

Problem:

Motivation: Compare the following two polynomial parametric approximants of degree 5 for approximation of the unit circle:

$$\begin{pmatrix} 1 - \frac{1}{2}t^2 + \frac{1}{24}t^4 \\ t - \frac{1}{6}t^3 + \frac{1}{120}t^5 \end{pmatrix}; \quad \begin{pmatrix} 1 - (3+\sqrt{5})t^2 + (1+\sqrt{5})t^4 \\ (1+\sqrt{5})t - (3+\sqrt{5})t^3 + t^5 \end{pmatrix}.$$



Problem:

- The implicit equations of the unit circle and the unit hyperbola are
$$x^2 \pm y^2 = 1.$$

- Task: find two nonconstant polynomials $x_n, y_n \in \mathbb{R}[t]$ of degree $\leq n$, such that
$$x_n^2(t) + y_n^2(t) = 1 + t^{2n} \quad (1)$$
for the elliptic case and
$$x_n^2(t) - y_n^2(t) = 1 \pm t^{2n} \quad (2)$$
for the hyperbolic case.

Problem:

Our aim:

- find all possible solutions for equations (1) and (2),
- precisely analyse the best solution,
- show, that the error of this best approximant decreases exponentially with the growing degree n.

Conic sections:

- By choosing an appropriate coordinate system, ellipse and hyperbola can be written as

$$\left(\frac{x - x_0}{a}\right)^2 \pm \left(\frac{y - y_0}{b}\right)^2 = 1.$$

- By a translation and scaling we can further obtain

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- We have to find two nonconstant polynomials x_n in y_n, of degree $\leq n$, such that

$$x_n^2(t) \pm y_n^2(t) = 1 + \varepsilon(t). \quad (3)$$

Conic sections:

- Since we will interpolate one point and tangent direction at this point:

$$x_n(0) = 1, \quad x_n'(0) = 0, \quad y_n(0) = 0, \quad y_n'(0) = 1.$$

- A residual polynomial ε is of degree at most $2n$. To have the approximation error as small as possible in the vicinity of the interpolation point, ε should be spanned by t^{2n} only. Thus

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- Polynomials x_n and y_n are now of the form

$$x_n(t) := 1 + \sum_{\ell=2}^{n} a_\ell\, t^\ell, \quad y_n(t) := t + \sum_{\ell=2}^{n} b_\ell\, t^\ell,$$

which gives

$$\left(1 + \sum_{\ell=2}^{n} a_\ell\, t^\ell\right)^2 \pm \left(t + \sum_{\ell=2}^{n} b_\ell\, t^\ell\right)^2 = 1 + \left(a_n^2 \pm b_n^2\right)\, t^{2n}.$$

Conic sections:

- Let
$$A := \frac{1}{\sqrt[2n]{|a_n^2 \pm b_n^2|}}.$$
Linear scaling of the parameter $t \mapsto t/A$ and introduction of new variables
$$\alpha_\ell := a_\ell\, A^\ell, \quad \beta_\ell := b_\ell\, A^\ell, \quad \ell = 1, 2, \ldots, n,$$
where $a_1 := 0, b_1 := 1$, transform the problem into the problem of finding
$$x_n(t) := 1 + \sum_{\ell=2}^{n} \alpha_\ell\, t^\ell, \quad y_n(t) := \sum_{\ell=1}^{n} \beta_\ell\, t^\ell, \quad \beta_1 > 0;$$
$$x_n^2(t) \pm y_n^2(t) = 1 + \mathrm{sign}(a_n^2 \pm b_n^2)\, t^{2n}. \tag{4}$$
- Elliptic case: one possibility.
- Hyperbolic case: two possibilities.

Solutions for the elliptic case:

- The equation $x_n^2(t) + y_n^2(t) = 1 + t^{2n}$ can be rewritten as

$$(x_n(t) + \mathrm{i}\, y_n(t))\,(x_n(t) - \mathrm{i}\, y_n(t)) = \prod_{k=0}^{2n-1} \left(t - e^{\mathrm{i}\,\frac{2k+1}{2n}\pi}\right). \qquad (5)$$

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- From the uniqueness of the polynomial factorization over $\mathbb{C}$ up to a constant factor, and from the fact that the factors in (5) appear in conjugate pairs, it follows

$$x_n(t) + \mathrm{i}\, y_n(t) = \gamma \prod_{k=0}^{n-1} \left(t - e^{\mathrm{i}\,\sigma_k \frac{2k+1}{2n}\pi}\right), \quad \gamma \in \mathbb{C}, \quad |\gamma| = 1,$$

where $\sigma_k = \pm 1$.

Solutions for the elliptic case:

- In order to interpolate the point $(1, 0)$:

$$\gamma := (-1)^n \prod_{k=0}^{n-1} e^{-\mathrm{i}\,\sigma_k \frac{2k+1}{2n}\pi}.$$

- $$x_n(t) + \mathrm{i}\, y_n(t) = (-1)^n \prod_{k=0}^{n-1} \left(t\, e^{-\mathrm{i}\,\sigma_k \frac{2k+1}{2n}\pi} - 1 \right) =: p_e(t; \boldsymbol{\sigma}),$$

  where $\boldsymbol{\sigma} = (\sigma_k)_{k=0}^{n-1} \in \{-1, 1\}^n$.

- Therefore we have 2^n solutions.
- We have to eliminate those with z $\beta_1 = 0$.
- The remaining ones appear in pairs $(x_n, \pm y_n)$, thus precisely half of them fulfill the requirement $\beta_1 > 0$.

Solutions for the hyperbolic case:

- Our equation:

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- For $a_n^2 < b_n^2$:

$$(x_n(t) + y_n(t))\,(x_n(t) - y_n(t)) = 1 - t^{2n} = (1 - t^2)\prod_{k=1}^{n-1}\left(t^2 - 2\cos\left(\frac{k\pi}{n}\right)t + 1\right). \qquad (6)$$

- For $a_n^2 > b_n^2$:

$$(x_n(t) + y_n(t))\,(x_n(t) - y_n(t)) = 1 + t^{2n} = \prod_{k=0}^{n-1}\left(t^2 - 2\cos\left(\frac{2k+1}{2n}\pi\right)t + 1\right). \qquad (7)$$

Solutions for the hyperbolic case:

- The right sides of equations (6) and (7) can be written as a product of two polynomials p_h and q_h. Then we can take

$$x_n(t) = \frac{1}{2}(p_h(t) + q_h(t)), \quad y_n(t) = \pm\frac{1}{2}(p_h(t) - q_h(t)).$$

- Since x_n and y_n are of degree $\leq n$, also both p_h and q_h must be of degree n.

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- In the case $a_n^2 < b_n^2$, we take

$$p_h(t) := p_h(t; \mathcal{I}_n) :=$$

$$(1+t)^{\frac{1-(-1)^n}{2}} \prod_{\substack{k \in \mathcal{I}_n \subseteq \{1,2,\ldots,n-1\} \\ |\mathcal{I}_n| = \lfloor \frac{n}{2} \rfloor}} \left(t^2 - 2\cos\left(\frac{k\pi}{n}\right) t + 1 \right). \quad (8)$$

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- In the case $a_n^2 > b_n^2$ the solutions exist only for even n and we take

$$p_h(t) := p_h(t; \mathcal{I}_n) := \prod_{\substack{k \in \mathcal{I}_n \subseteq \{1,2,\ldots,n-1\} \\ |\mathcal{I}_n| = \frac{n}{2}}} \left(t^2 - 2\cos\left(\frac{2k+1}{2n}\pi\right) t + 1\right).$$

Solutions for the hyperbolic case:

- Again we have to eliminate solutions with $\beta_1 = 0$ and from the remaining pairs $(x_n, \pm y_n)$ select only those with $\beta_1 > 0$.

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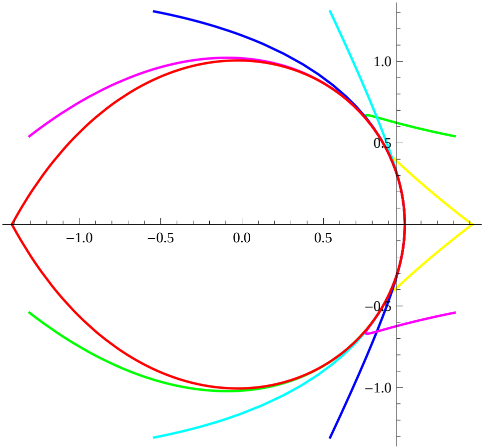
The number of admissible solutions grows exponentially with n:

n	2	3	4	5	6	7	8	9	10
ellipt. case	1	3	6	15	27	64	120	254	495
hyper. $a_n^2 < b_n^2$	0	1	2	5	8	20	32	70	120
hyper. $a_n^2 > b_n^2$	1	0	2	0	9	0	32	0	125

Solutions of the problem:

The example of all admissible solutions for the elliptic case for $n = 4$:



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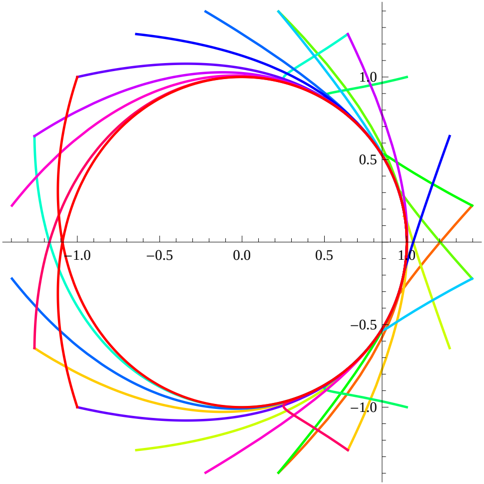
The example of all admissible solutions for the elliptic case for $n = 5$:



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Theorem:

The best solution for the elliptic case is

$$x_n(t) = \mathrm{Re}\left(p_e\left(t;\boldsymbol{\sigma}^*\right)\right), \quad y_n(t) = \mathrm{Im}\left(p_e\left(t;\boldsymbol{\sigma}^*\right)\right), \quad \boldsymbol{\sigma}^* = (1)_{k=0}^{n-1}.$$

The best solution for the hyperbolic case is

$$x_n(t) = \frac{1}{2}\left(p_h\left(t;\mathcal{I}_n^*\right) + p_h\left(-t;\mathcal{I}_n^*\right)\right), \; y_n(t) = \frac{1}{2}\left(p_h\left(t;\mathcal{I}_n^*\right) - p_h\left(-t;\mathcal{I}_n^*\right)\right),$$

where p_h is defined in (8) for odd n and in (9) for even n, and

$$\mathcal{I}_n^* = \left\{\left\lfloor \frac{n+1}{2} \right\rfloor, \left\lfloor \frac{n+1}{2} \right\rfloor + 1, \ldots, n-1\right\}.$$

In all cases: $\quad \beta_1 = \dfrac{1}{\omega_n}, \quad \omega_n := \sin \dfrac{\pi}{2n}.$

Best solution:

- Any solution for the elliptic case, for which x_n is an even and y_n an odd function, can be transformed into a solution for the hyperbolic case by using the map

$$\begin{aligned} x_n(t) &\mapsto x_n(\mathrm{i}\, t), \\ y_n(t) &\mapsto -\mathrm{i}\, y_n(\mathrm{i}\, t). \end{aligned} \tag{10}$$

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$$y_n(t) \mapsto -\mathrm{i}\; y_n(\mathrm{i}\, t).$$

- The coefficient β_1 is equal for both best solutions, and it is preserved by the map (10).
- Therefore: If we can prove that in the elliptic case the polynomial x_n is an even and y_n an odd function, then the best solution for the elliptic case is mapped by (10) to the best solution for the hyperbolic case.

Best solution:

Theorem:

The coefficients of the best solution in the elliptic case are obtained as

$$\alpha_k = \begin{cases} \sum_{j=0}^{k(n-k)} P(j,k,n-k)\cos\left(\frac{k^2}{2n}\pi + \frac{j}{n}\pi\right), & k \text{ is even}, \\ 0, & k \text{ is odd}, \end{cases}$$

$$\beta_k = \begin{cases} 0, & k \text{ is even}, \\ \sum_{j=0}^{k(n-k)} P(j,k,n-k)\sin\left(\frac{k^2}{2n}\pi + \frac{j}{n}\pi\right), & k \text{ is odd}, \end{cases}$$

- $P(j,k,r)$ denotes the number of integer partitions of $j \in \mathbb{N}$ with $\leq k$ parts, all between 1 and r, where $k, r \in \mathbb{N}$,
- $P(0,k,r) := 1$.

Best solution:

Corollary:

- For the best solution (in both cases), the polynomial x_n is an even function, and y_n is an odd one.
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Corollary:

For the coefficients of the polynomials x_n and y_n in the elliptic case it holds

$$\alpha_{n-k} + \mathrm{i}\,\beta_{n-k} = \mathrm{i}^n(\alpha_k - \mathrm{i}\,\beta_k), \quad k = 0, 1, \ldots, \lfloor n/2 \rfloor.$$

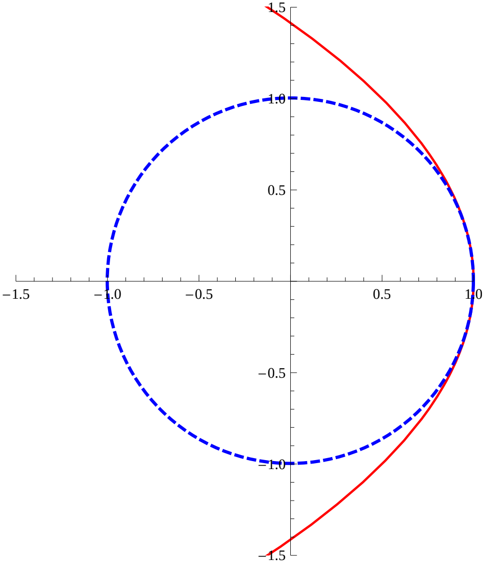
Moreover:

$$\alpha_k = \pm\beta_{n-k}, \quad \text{for } n = 4\ell \pm 1,$$
$$\alpha_{n-k} = \mp\alpha_k, \quad \beta_{n-k} = \pm\beta_k, \quad \text{for } n = 4\ell + 1 \pm 1.$$

Best solution:

n	$x_n(t), \quad y_n(t)$
2	$x_2(t) = 1 \mp t^2, \quad y_2(t) = \sqrt{2}\, t$
3	$x_3(t) = 1 \mp 2\, t^2, \quad y_3(t) = 2\, t \mp t^3$
4	$x_4(t) = 1 \mp (2+\sqrt{2})t^2 + t^4$ $y_4(t) = \sqrt{4+2\sqrt{2}}(t \mp t^3)$
5	$x_5(t) = 1 \mp (3+\sqrt{5})t^2 + (1+\sqrt{5})t^4$ $y_5(t) = (1+\sqrt{5})t \mp (3+\sqrt{5})t^3 + t^5$
6	$x_6(t) = 1 \mp 2(2+\sqrt{3})t^2 + 2(2+\sqrt{3})t^4 \mp t^6$ $y_6(t) = (\sqrt{2}+\sqrt{6})t \mp \sqrt{2}\,(3+2\sqrt{3})t^3 + (\sqrt{2}+\sqrt{6})t^5$

Best solution:



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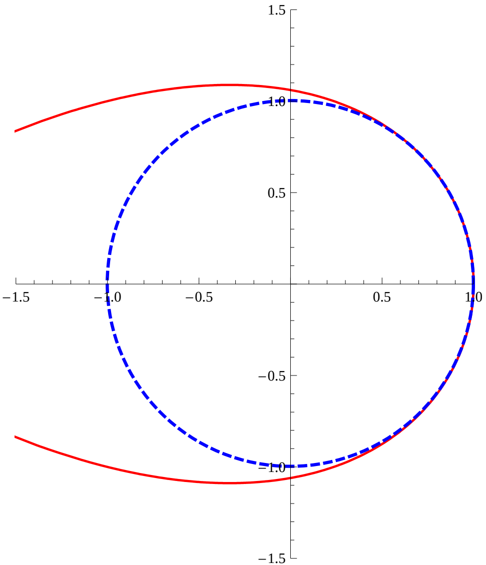

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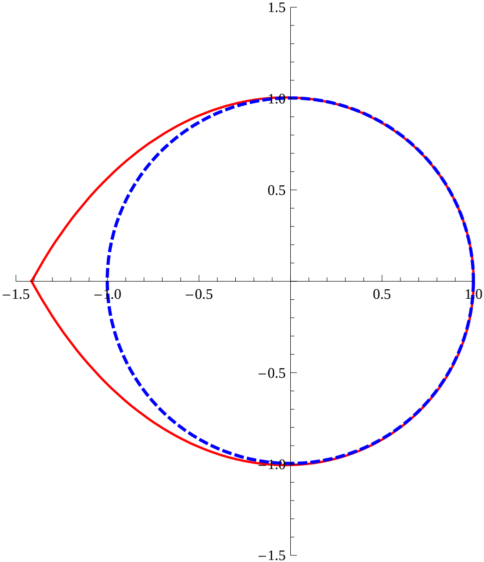

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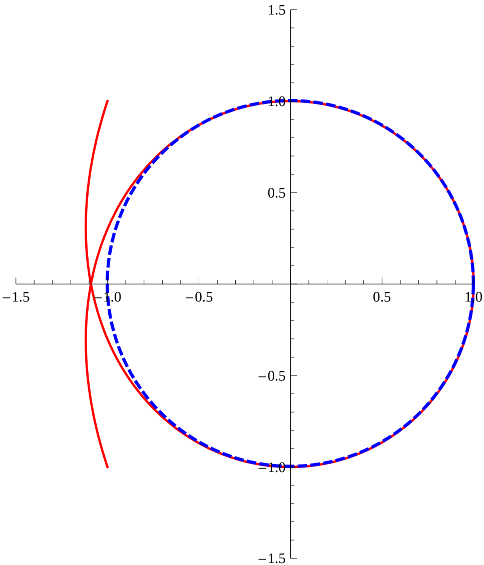

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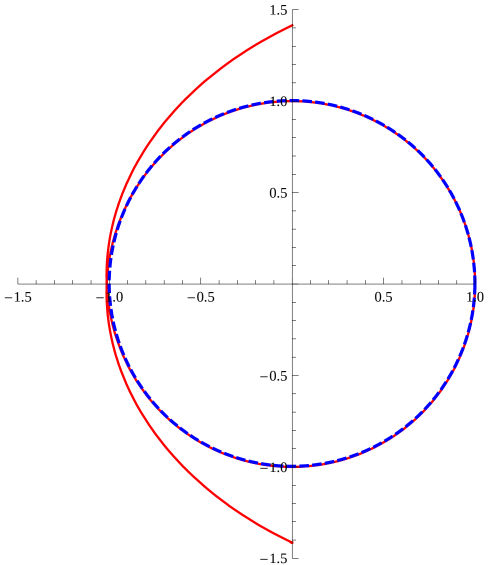

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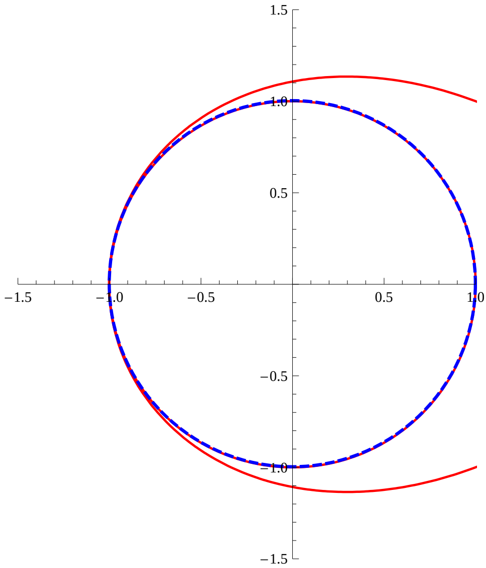
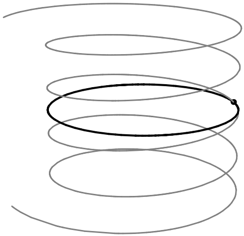


Error analysis - elliptic case:

- The number of winds of polynomial curve around the origin is $\lfloor \frac{n}{4} \rfloor$.



Error analysis - elliptic case:

Radial distance of the segment of the polynomial curve

$$(x_n(t), y_n(t))^T ,$$

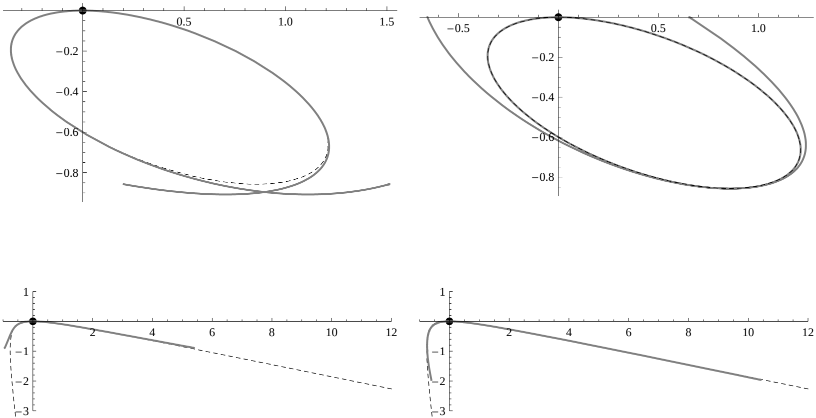
which approximates the unit circle, can be expressed as

$$\left(\frac{\pi^2}{2n}\right)^{2n} + \mathcal{O}\left(\left(\frac{\pi^2}{2n}\right)^{2n+1}\right).$$

Error analysis - elliptic case:

	ellip. case ($s \in [-\pi, \pi]$)	
n	h_n	error
4	1	0.41421
5	0.84612	0.08999
6	0.74225	0.01389
7	0.65658	0.00138
8	0.58526	$9.5 \cdot 10^{-5}$
9	0.52643	$4.8 \cdot 10^{-6}$
10	0.47766	$1.9 \cdot 10^{-7}$
11	0.43680	$6.1 \cdot 10^{-9}$
12	0.40217	$1.6 \cdot 10^{-10}$
13	0.37249	$3.5 \cdot 10^{-12}$
14	0.34681	$6.6 \cdot 10^{-14}$
15	0.32438	$1.1 \cdot 10^{-15}$

Generalizations:



Thank you!