Graphs of separability at most two:
structural characterizations and their consequences

Ferdinando Cicalese[1] Martin Milanič[2]

[1]DIA, University of Salerno, Fisciano, Italy

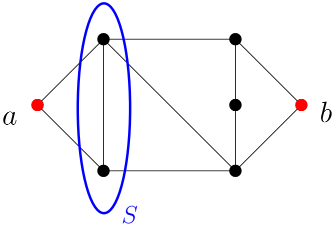
[2]FAMNIT in PINT, Univerza na Primorskem

Raziskovalni matematični seminar, FAMNIT, 18. oktober 2010

Separators and separability

G - a (simple, finite, undirected) graph
a,b - two vertices of G

An (a, b)-separator is a set $S \subseteq V(G)$ such that a and b are in different connected components of $G - S$.

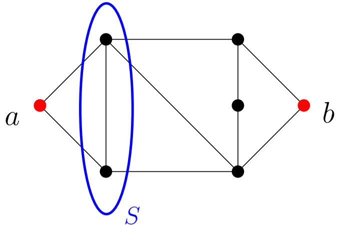


Separability of $\{a, b\}$: the smallest size of an (a, b)-separator.

Separators and separability

G - a (simple, finite, undirected) graph
a,b - two vertices of G

An (a, b)-separator is a set $S \subseteq V(G)$ such that a and b are in different connected components of $G - S$.

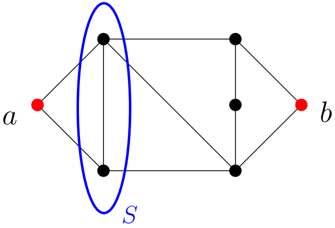


Separability of $\{a, b\}$: the smallest size of an (a, b)-separator.

Separators and separability

G - a (simple, finite, undirected) graph
a,b - two vertices of G

An (a, b)-separator is a set $S \subseteq V(G)$ such that a and b are in different connected components of $G - S$.

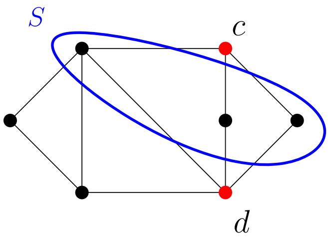


separability$(a, b) = 2$

Separators and separability

G - a (simple, finite, undirected) graph
a,b - two vertices of G

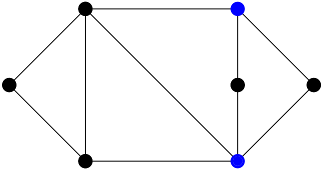
An (a, b)-separator is a set $S \subseteq V(G)$ such that a and b are in different connected components of $G - S$.



separability$(c, d) = 3$

Separability of graphs

The separability of a graph G is the maximum over all separabilities of non-adjacent vertex pairs...

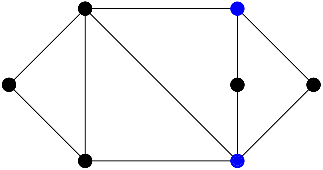


a graph of separability 3

... unless G is complete, in which case we define $separability(G) = 0$.

Separability of graphs

The separability of a graph G is the maximum over all separabilities of non-adjacent vertex pairs...



a graph of separability 3

... unless G is complete, in which case we define $separability(G) = 0$.

Menger's Theorem and separability

By Menger's Theorem,

$$\begin{aligned} \textit{separability}(a,b) &= \textit{min size of an } (a,b)\textit{-separator} \\ &= \textit{max } \# \textit{ internally vertex-disjoint } (a,b)\textit{-paths}. \end{aligned}$$

Therefore, for a non-complete graph G,

separability(G) = max # *internally vertex-disjoint paths connecting two non-adjacent vertices in* G.

Menger's Theorem and separability

By Menger's Theorem,

$$\begin{aligned} \textit{separability}(a,b) &= \textit{min size of an } (a,b)\textit{-separator} \\ &= \textit{max } \# \textit{ internally vertex-disjoint } (a,b)\textit{-paths}. \end{aligned}$$

Therefore, for a non-complete graph G,

separability(G) = max # *internally vertex-disjoint paths connecting two non-adjacent vertices in* G.

Graphs of bounded separability

For $k \geq 0$, let

$$\mathcal{G}_k = \{G : \mathit{separability}(G) \leq k\}\,.$$

Graphs in $\mathcal{G}_k$:

- generalize graphs of maximum degree k,
- generalize pairwise k-separable graphs,

  G.L. Miller, Isomorphism of graphs which are pairwise k-separable. *Informat. and Control* 56 (1983) 21–33.
- are related to the parsimony haplotyping problem from computational biology.

Graphs of bounded separability

For $k \geq 0$, let

$$\mathcal{G}_k = \{G : separability(G) \leq k\}\,.$$

Graphs in $\mathcal{G}_k$:

- generalize graphs of maximum degree k,
- generalize pairwise k-separable graphs,

  G.L. Miller, Isomorphism of graphs which are pairwise k-separable. *Informat. and Control* 56 (1983) 21–33.

- are related to the parsimony haplotyping problem from computational biology.

Graphs of bounded separability

For $k \geq 0$, let

$$\mathcal{G}_k = \{G : separability(G) \leq k\}.$$

Graphs in $\mathcal{G}_k$:

- generalize graphs of maximum degree k,
- generalize pairwise k-separable graphs,

  G.L. Miller, Isomorphism of graphs which are pairwise k-separable. *Informat. and Control* 56 (1983) 21–33.
- are related to the parsimony haplotyping problem from computational biology.

Graphs of bounded separability

For $k \geq 0$, let

$$\mathcal{G}_k = \{G : separability(G) \leq k\}\,.$$

Graphs in $\mathcal{G}_k$:

- generalize graphs of maximum degree k,
- generalize pairwise k-separable graphs,

  G.L. Miller, Isomorphism of graphs which are pairwise k-separable. *Informat. and Control* 56 (1983) 21–33.

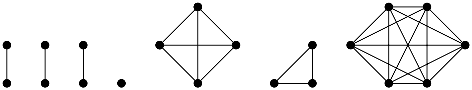
- are related to the parsimony haplotyping problem from computational biology.

The main question

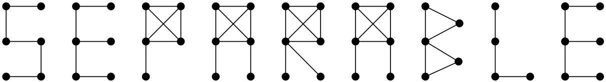
Can we characterize graphs of separability at most k,
at least for small values of k?

Structure of graphs in $\mathcal{G}_0$ and $\mathcal{G}_1$

Graphs of separability 0 $=$ disjoint unions of complete graphs



Graphs of separability at most 1 $=$ *block graphs*: graphs every block of which is complete.



Outline

$\mathcal{G}_2$, graphs of separability at most 2:

- generalize complete graphs, trees, cycles, block-cactus graphs
- characterizations
- algorithmic and complexity results

Graphs in $\mathcal{G}_k$:

- connection to the parsimony haplotyping problem

Outline

$\mathcal{G}_2$, graphs of separability at most 2:

- generalize complete graphs, trees, cycles, block-cactus graphs
- characterizations
- algorithmic and complexity results

Graphs in $\mathcal{G}_k$:

- connection to the parsimony haplotyping problem

Characterizations

A structure theorem

Complete graphs, cycles are in $\mathcal{G}_2$.

A structure theorem

Complete graphs, cycles are in $\mathcal{G}_2$.

Theorem

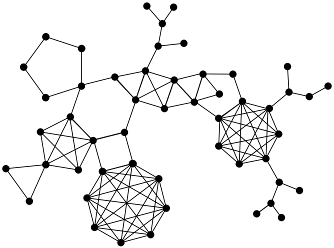
A connected graph G is in $\mathcal{G}_2$ if and only if G arises from complete graphs and cycles by pasting along vertices or edges.

A structure theorem

Complete graphs, cycles are in $\mathcal{G}_2$.

Theorem

A connected graph G is in $\mathcal{G}_2$ if and only if G arises from complete graphs and cycles by pasting along vertices or edges.

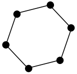


A structure theorem

Complete graphs, cycles are in $\mathcal{G}_2$.

Theorem

A connected graph G is in $\mathcal{G}_2$ if and only if G arises from complete graphs and cycles by pasting along vertices or edges.

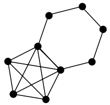


A structure theorem

Complete graphs, cycles are in $\mathcal{G}_2$.

Theorem

A connected graph G is in $\mathcal{G}_2$ if and only if G arises from complete graphs and cycles by pasting along vertices or edges.

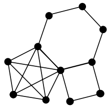


A structure theorem

Complete graphs, cycles are in $\mathcal{G}_2$.

Theorem

A connected graph G is in $\mathcal{G}_2$ if and only if G arises from complete graphs and cycles by pasting along vertices or edges.

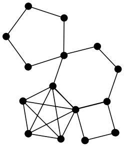


A structure theorem

Complete graphs, cycles are in $\mathcal{G}_2$.

Theorem

A connected graph G is in $\mathcal{G}_2$ if and only if G arises from complete graphs and cycles by pasting along vertices or edges.

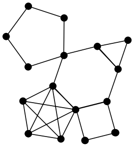


A structure theorem

Complete graphs, cycles are in $\mathcal{G}_2$.

Theorem

A connected graph G is in $\mathcal{G}_2$ if and only if G arises from complete graphs and cycles by pasting along vertices or edges.

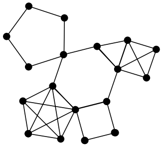


A structure theorem

Complete graphs, cycles are in $\mathcal{G}_2$.

Theorem

A connected graph G is in $\mathcal{G}_2$ if and only if G arises from complete graphs and cycles by pasting along vertices or edges.

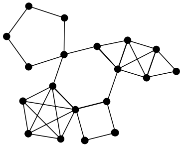


A structure theorem

Complete graphs, cycles are in $\mathcal{G}_2$.

Theorem

A connected graph G is in $\mathcal{G}_2$ if and only if G arises from complete graphs and cycles by pasting along vertices or edges.

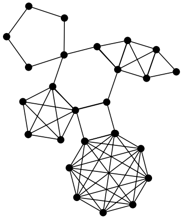


A structure theorem

Complete graphs, cycles are in $\mathcal{G}_2$.

Theorem

A connected graph G is in $\mathcal{G}_2$ if and only if G arises from complete graphs and cycles by pasting along vertices or edges.

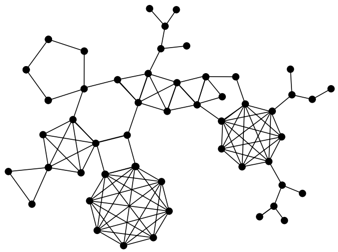


A structure theorem

Complete graphs, cycles are in $\mathcal{G}_2$.

Theorem

A connected graph G is in $\mathcal{G}_2$ if and only if G arises from complete graphs and cycles by pasting along vertices or edges.



Some consequences of the structure result

Corollary

Every graph in $\mathcal{G}_2$ contains either a simplicial vertex or two adjacent vertices of degree 2.

$v \in V(G)$ is *simplicial* if its neighborhood is a clique.

Corollary

Graphs in $\mathcal{G}_2$ are χ-bounded:
There exists a function f such that for every $G \in \mathcal{G}_2$,

$$\chi(G) \leq f(\omega(G)) .$$

Some consequences of the structure result

Corollary

Every graph in $\mathcal{G}_2$ contains either a simplicial vertex or two adjacent vertices of degree 2.

$v \in V(G)$ is *simplicial* if its neighborhood is a clique.

Corollary

Graphs in $\mathcal{G}_2$ are χ-bounded:
There exists a function f such that for every $G \in \mathcal{G}_2$,

$$\chi(G) \leq f(\omega(G)) .$$

Some consequences of the structure result

Is tree-width of graphs in $\mathcal{G}_2$ bounded by a constant?

No $\because$ complete graphs are in $\mathcal{G}_2$.

Corollary

For every $G \in \mathcal{G}_2$,

$$tw(G) \leq \max\{2, \omega(G) - 1\}\,.$$

(This is best possible: no similar result holds for $\mathcal{G}_3$.)

Some consequences of the structure result

Is tree-width of graphs in $\mathcal{G}_2$ bounded by a constant?

No $\because$ complete graphs are in $\mathcal{G}_2$.

Corollary

For every $G \in \mathcal{G}_2$,

$$tw(G) \leq \max\{2, \omega(G) - 1\}.$$

(This is best possible: no similar result holds for $\mathcal{G}_3$.)

Some consequences of the structure result

Is tree-width of graphs in $\mathcal{G}_2$ bounded by a constant?

No $\because$ complete graphs are in $\mathcal{G}_2$.

Corollary

For every $G \in \mathcal{G}_2$,

$$tw(G) \leq \max\{2, \omega(G) - 1\}.$$

(This is best possible: no similar result holds for $\mathcal{G}_3$.)

Characterization by forbidden induced subgraphs

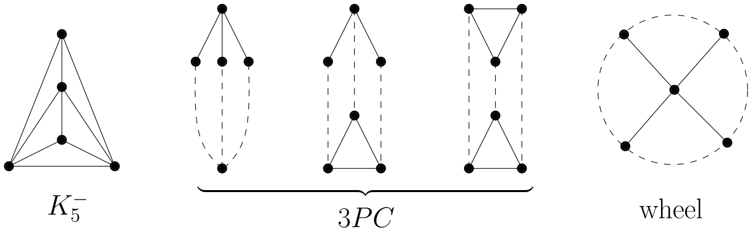
induced minor of G: a graph obtained from G by vertex deletions

Characterization by forbidden induced subgraphs

induced minor of G: a graph obtained from G by vertex deletions

Theorem

$\mathcal{G}_2$ = \{K_5^-, 3PC, wheels\}-induced-subgraph-free graphs.



Characterization by forbidden induced minors

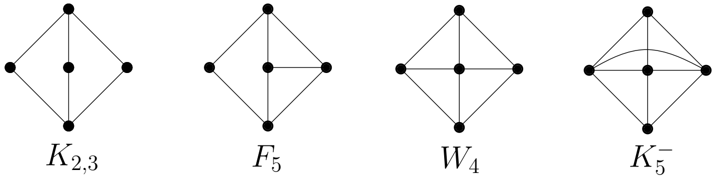
induced minor of G: a graph obtained from G by vertex deletions and edge contractions

Characterization by forbidden induced minors

induced minor of G: a graph obtained from G by vertex deletions and edge contractions

Theorem

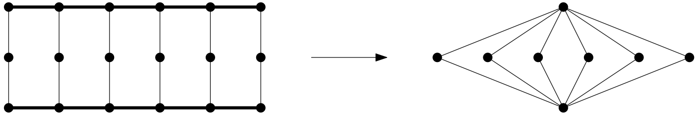
$\mathcal{G}_2 = \{K_{2,3}, F_5, W_4, K_5^-\}$*-induced-minor-free graphs.*



This is best possible

Theorem

$\mathcal{G}_k$ is closed under induced minors if and only if $k \leq 2$.



a graph from $\mathcal{G}_3$ contracted to a graph of separability 6

Algorithms and complexity

Good news

Some problems are solvable in polynomial time for graphs in $\mathcal{G}_k$, for every k:

- recognition
  - $O(|V(G)|^2)$ max flow computations
- finding a maximum weight clique
  - polynomially many maximal cliques

Good news

Some problems are solvable in polynomial time for graphs in $\mathcal{G}_k$, for every k:

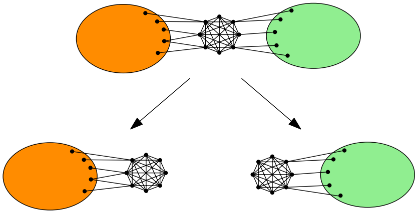
- recognition
  - $O(|V(G)|^2)$ max flow computations
- finding a maximum weight clique
  - polynomially many maximal cliques

Good news

For graphs in $\mathcal{G}_2$, the structure theorem leads to polytime algorithms for:

- maximum weight independent set (NP-hard for $\mathcal{G}_3$)
- coloring (NP-hard for $\mathcal{G}_3$)

The algorithms are based on the decomposition by clique separators.



Whitesides 1981, Tarjan 1985

Not so good news

Is clique-width of graphs in $\mathcal{G}_2$ bounded by a constant?

Proposition

Graphs in $\mathcal{G}_2$ are of unbounded clique-width.

Proposition

The following problems are NP-complete:

- *The dominating set problem for graphs in $\mathcal{G}_2$.*
- *The simple max cut problem for graphs in $\mathcal{G}_2$.*
- *The 3-colorability problem for graphs in $\mathcal{G}_3$ (planar, of maximum degree 6).*

Not so good news

Is clique-width of graphs in $\mathcal{G}_2$ bounded by a constant?

Proposition

Graphs in $\mathcal{G}_2$ are of unbounded clique-width.

Proposition

The following problems are NP-complete:

- *The dominating set problem for graphs in $\mathcal{G}_2$.*
- *The simple max cut problem for graphs in $\mathcal{G}_2$.*
- *The 3-colorability problem for graphs in $\mathcal{G}_3$ (planar, of maximum degree 6).*

Not so good news

Is clique-width of graphs in $\mathcal{G}_2$ bounded by a constant?

Proposition

Graphs in $\mathcal{G}_2$ are of unbounded clique-width.

Proposition

The following problems are NP-complete:

- *The dominating set problem for graphs in $\mathcal{G}_2$.*
- *The simple max cut problem for graphs in $\mathcal{G}_2$.*
- *The 3-colorability problem for graphs in $\mathcal{G}_3$ (planar, of maximum degree 6).*

Not so good news

Is clique-width of graphs in $\mathcal{G}_2$ bounded by a constant?

Proposition

Graphs in $\mathcal{G}_2$ are of unbounded clique-width.

Proposition

The following problems are NP-complete:

- *The dominating set problem for graphs in $\mathcal{G}_2$.*
- *The simple max cut problem for graphs in $\mathcal{G}_2$.*
- *The 3-colorability problem for graphs in $\mathcal{G}_3$ (planar, of maximum degree 6).*

Not so good news

Is clique-width of graphs in $\mathcal{G}_2$ bounded by a constant?

Proposition

Graphs in $\mathcal{G}_2$ are of unbounded clique-width.

Proposition

The following problems are NP-complete:

- *The dominating set problem for graphs in $\mathcal{G}_2$.*
- *The simple max cut problem for graphs in $\mathcal{G}_2$.*
- *The 3-colorability problem for graphs in $\mathcal{G}_3$ (planar, of maximum degree 6).*

Connection to the parsimony haplotyping problem

A problem from computational biology

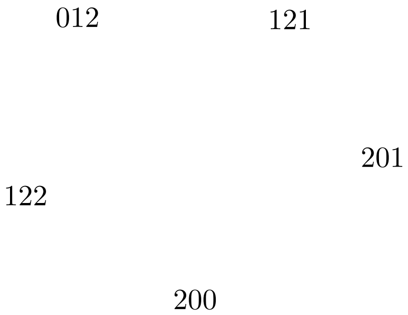
Parsimony Haplotyping:

Given: a set of n vectors in $\{0,1,2\}^m$ (*genotypes*).

Task: find the minimum size of a set of vectors in $\{0,1\}^m$ (*haplotypes*) such that every genotype can be expressed as the sum of two haplotypes from the set.

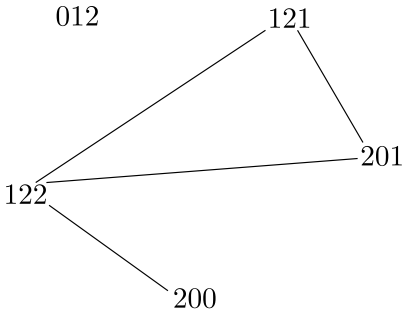
Addition rules: $0+0=0$, $1+1=1$, $0+1=1+0=2$

Compatibility graph G: the graph with
$V(G) = \{\text{genotypes}\}$ and
$E(G) = \{gg' \ : \ \nexists r \text{ such that } \{g_r, g'_r\} = \{0,1\}\}$.



A problem from computational biology

Compatibility graph G: the graph with $V(G) = \{\text{genotypes}\}$ and $E(G) = \{gg' \ : \ \nexists r \text{ such that } \{g_r, g'_r\} = \{0,1\}\}$.



Connection to separability

A parsimony haplotyping instance is *k-bounded* if in every coordinate, at most k genotypes contain a 2.

Theorem

$\mathcal{G}_k = \{$*compatibility graphs of k-bounded PH instances*$\}$.

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$
PARSIMONY HAPLOTYPING	polynomial	**?**	NP-complete

van Iersel–Keijsper–Kelk–Stougie 2008

Sharan–Halldórsson–Istrail 2006

Connection to separability

A parsimony haplotyping instance is *k-bounded* if in every coordinate, at most k genotypes contain a 2.

Theorem

$\mathcal{G}_k = \{$*compatibility graphs of k-bounded PH instances*$\}$.

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$
PARSIMONY HAPLOTYPING	polynomial	**?**	NP-complete

van Iersel–Keijsper–Kelk–Stougie 2008

Sharan–Halldórsson–Istrail 2006

Connection to separability

A parsimony haplotyping instance is *k-bounded* if in every coordinate, at most k genotypes contain a 2.

Theorem

$\mathcal{G}_k = \{$*compatibility graphs of k-bounded PH instances*$\}$.

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$
PARSIMONY HAPLOTYPING	polynomial	**?**	NP-complete

van Iersel–Keijsper–Kelk–Stougie 2008

Sharan–Halldórsson–Istrail 2006

Some open problems

For $k \geq 3$, characterize graphs in $\mathcal{G}_k$ in terms of:

- forbidden induced subgraphs,
- decomposition properties.

For $k \geq 3$, determine whether graphs in $\mathcal{G}_k$ are χ-bounded.

Determine the complexity of:

- the independent domination problem for graphs in $\mathcal{G}_2$,
- 2-bounded parsimony haplotyping.

Some open problems

For $k \geq 3$, characterize graphs in $\mathcal{G}_k$ in terms of:

- forbidden induced subgraphs,
- decomposition properties.

For $k \geq 3$, determine whether graphs in $\mathcal{G}_k$ are χ-bounded.

Determine the complexity of:

- the independent domination problem for graphs in $\mathcal{G}_2$,
- 2-bounded parsimony haplotyping.

Some open problems

For $k \geq 3$, characterize graphs in $\mathcal{G}_k$ in terms of:

- forbidden induced subgraphs,
- decomposition properties.

For $k \geq 3$, determine whether graphs in $\mathcal{G}_k$ are χ-bounded.

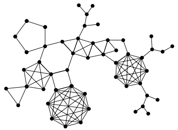
Determine the complexity of:

- the independent domination problem for graphs in $\mathcal{G}_2$,
- 2-bounded parsimony haplotyping.

Conclusion

Separators of size at most 2 sometimes help...

- decomposition along separating cliques of size at most two into cycles and complete graphs,
- $tw(G) \leq f(\omega(G))$,
- $\chi(G) \leq f(\omega(G))$.



...but not always:

- dominating set is NP-complete,
- simple max cut is NP-complete,
- clique-width is unbounded.

The end

HVALA ZA POZORNOST