A problem from the theory of distance-regular graphs
Part II

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Notation

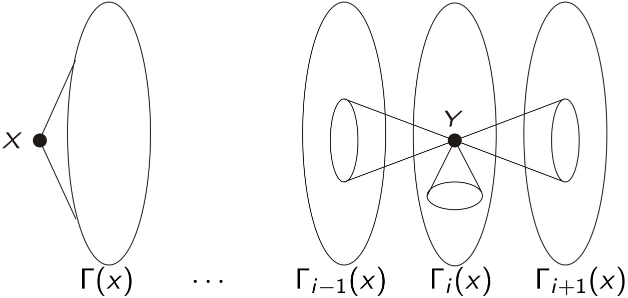
- Γ - finite, connected simple graph
- X - vertex set of Γ
- $d(x, y)$ - distance between $x, y \in X$
- D - diameter of Γ
- $\Gamma_i(x) = \{y \in X \mid d(x, y) = i\}$ $(\Gamma(x) := \Gamma_1(x))$

Distance partition

Pick $x \in X$ and let $D(x)$ denote the diameter of Γ wrt x. Then

$$\{\Gamma_0(x), \Gamma_1(x), \ldots, \Gamma_{D(x)}(x)\}$$

is a *distance partition of X wrt x*.



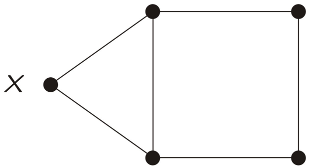
Distance-regularity with respect to a vertex

Pick $x \in X$. Graph Γ is *distance-regular wrt x*, if for $0 \leq i \leq D(x)$ there exist nonnegative integers $a_i(x)$, $b_i(x)$ in $c_i(x)$, such that for every $y \in \Gamma_i(x)$ we have

$$|\Gamma_{i-1}(x) \cap \Gamma(y)| = c_i(x)$$
$$|\Gamma_i(x) \cap \Gamma(y)| = a_i(x)$$
$$|\Gamma_{i+1}(x) \cap \Gamma(y)| = b_i(x)$$

$c_i(x), a_i(x), b_i(x)$ - intersection numbers of Γ (wrt x).

Distance-regularity with respect to a vertex



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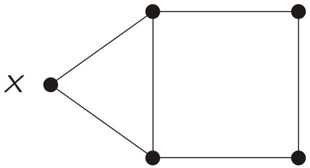

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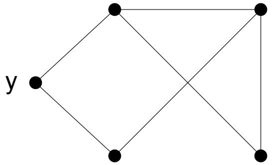
Note that we always have $c_0(x) = 0,\ c_1(x) = 1,\ a_0(x) = 0$ and $b_{D(x)} = 0$!

Distance-regularity with respect to a vertex



Not distance-regular with respect to y!

Distance-regular and distance-biregular graphs

Theorem

(Godsil & Shawe-Taylor) Assume that Γ is distance-regular wrt every $x \in X$. Then exactly one of the following (i), (ii) holds:

(i) *Diameter $D(x)$ and numbers $c_i(x) = c_i, a_i(x) = a_i$ in $b_i(x) = b_i$ are independent of the choice of x. In this case we call Γ* distance-regular.

(ii) *Graph Γ is bipartite, diameter $D(x)$ and numbers $c_i(x), a_i(x), b_i(x)$ depend only on bipartition set of x. In this case we call Γ* distance-biregular.

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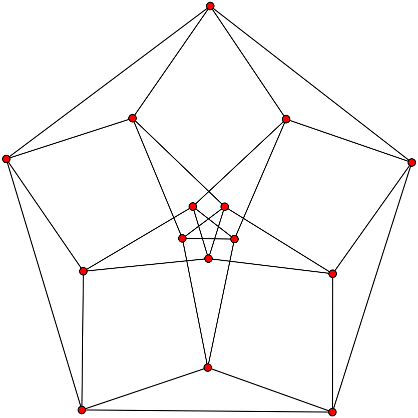
From now on we assume Γ is distance-regular!!!

Examples of drg

Line graph of Petersen graph:

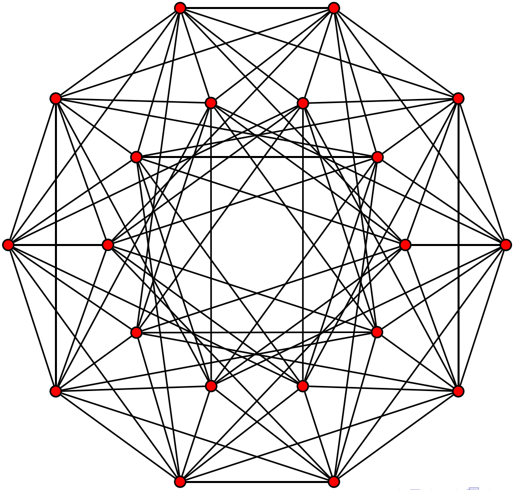


Examples of drg

Johnson graphs

Let S denote a finite set with n elements and pick $0 \leq e \leq n$. Johnson graph $J(n,e)$ has for vertices all subsets of S with e elements. Two vertices x, y are adjacent if and only if $|x \cap y| = e - 1$. Johnson graph $J(n,e)$ is distance-regular with $D = \min\{e, n-e\}$, $b_i = (e-i)(n-e-i)$, $c_i = i^2$.

Johnson graph $J(6,3)$

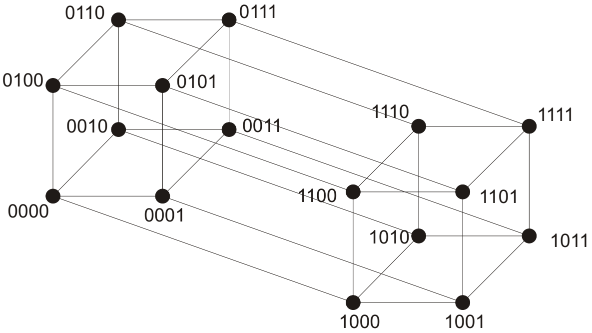


Examples of drg

Hamming graphs

For integers $n \geq 1$ and $q \geq 2$ let X denote the set of all sequences of length n and with elements in $\{0, 1, \ldots, q-1\}$. Hamming graph $H(n, q)$ has vertex set X. Two vertices x, y are adjacent if and only if they differ in exactly one coordinate. Hamming graph $H(n, q)$ is distance-regular with $D = n$, $b_i = (n-i)(q-1)$, $c_i = i$.

Hamming graph $H(4, 2)$



Examples of drg

Grassmann graphs

Let $\mathbb{F}$ denote a finite field with $|\mathbb{F}| = q$ and let V denote a n-dimensional vector space over $\mathbb{F}$. Pick $0 \le e \le n$. Grassmann graph $G(q, n, e)$ has for vertices all subspaces of V with dimension e. Two vertices x, y are adjacent if and only if $\dim(x \cap y) = e - 1$. Grassmann graph $G(q, n, e)$ is distance-regular with

$$D = \min\{e, n-e\}, \; b_i = q^{2i+1} \frac{q^{e-i}-1}{q-1} \frac{q^{n-e-i}-1}{q-1}, \; c_i = \left(\frac{q^i-1}{q-1}\right)^2.$$

Regularity

Theorem

Let Γ denote a distance-regular graph. Then:

(i) Γ *is regular with valency* $k = b_0$.

(ii) $c_i + a_i + b_i = k$ *for every* $0 \leq i \leq D$.

(iii) $a_i = k - c_i - b_i$ *for every* $0 \leq i \leq D$.

Cardinality of spheres

Theorem

Let Γ denote a distance-regular graph. Pick $x \in X$ and let $k_i = |\Gamma_i(x)| = |\{y \in X \mid d(x,y) = i\}|$ $(0 \leq i \leq D)$. Then $k_0 = 1$ and

$$k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} \qquad (1 \leq i \leq D)$$

Bipartite distance-regular graphs

Theorem

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Bipartite distance-regular graphs

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Let Γ denote a distance-regular graph. Then Γ is bipartite if and only if $a_i = 0$ for every $0 \leq i \leq D$.

We call Γ *almost bipartite* if $a_i = 0$ for $0 \leq i \leq D-1$ and $a_D \neq 0$.

Monotonicity

Theorem

Let Γ denote a distance-regular graph. Then:

(i) $b_0 > b_1 \geq b_2 \geq b_3 \geq \cdots \geq b_{D-1}$.

(ii) $c_1 \leq c_2 \leq \cdots \leq c_{D-1} \leq c_D$.

(iii) *If $i + j \leq D$ then $c_i \leq b_j$.*

Another (not so easy) example

Theorem

Let Γ denote a distance-regular graph with $b_0 \geq 3$. If $b_i = 1$ for some $i \leq D - 1$ then the following hold.

(i) *If $i + j \leq D$, then $c_j = 1$ and $c_{i+j} > a_j$.*

(ii) *If $2i \leq D$, then $c_{2i} > 1$.*

(iii) *If $2i + j \leq D$, then $a_j = 0$.*

(iv) $a_i \geq (c_2 - 1)a_{i+1} + a_1$.

Q-polynomial distance-regular graphs

- Purely algebraic definition (via some matrices associated with the adjacency matrix of Γ).
- Majority of known distance-regular graphs are Q-polynomial (Johnson graphs, Hamming graphs, Grassmann graphs, ...)
- Lot of nice combinatorial properties.

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Theorem (H. Lewis, 2000)

The girth of a Q-polynomial distance-regular graph is at most 6.

The main problem

Problem

Classify Q-polynomial distance-regular graphs with girth 6!

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Conjecture

Assume Γ is Q-polynomial distance-regular graphs with girth 6. Then Γ is either

- *generalized hexagon of order $(1, k-1)$, where $k = b_0$ is the valency of Γ, or*
- *the Odd graph on the set with cardinality $2D+1$, where D is the diameter of Γ.*

Girth 6

Assume from now on that Γ is Q-polynomial distance-regular graph with girth 6. Then $D \geq 3$, $a_1 = 0$, $c_2 = 1$ and $a_2 = 0$. If $k = 2$ then Γ is a cycle, therefore assume also $k \geq 3$.

Girth 6

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Theorem (Miklavic, 2004)

If $a_1 = a_2 = 0$, then Γ is either bipartite or almost bipartite.

Case $D = 3$

Theorem (Miklavic)

If $D = 3$, then Γ is either

- *generalized hexagon of order $(1, k-1)$, where $k = b_0$ is the valency of Γ (in this case Γ is bipartite), or*
- *the Odd graph on the set with cardinality 7 (in this case Γ is almost bipartite).*

Case $D \geq 4$, almost bipartite

Theorem (Lang, Terwilliger)

If $D \geq 4$ and Γ is almost bipartite, then Γ is the Odd graph on the set with cardinality $2D + 1$.

Case $D \geq 12$, bipartite

Theorem (Caughman 2004)

Assume Γ is bipartite with $D \geq 12$. Then exactly one of the following hold:

- *Γ is the Hamming graph $H(D, 2)$ - the D-dimensional hypercube;*
- *Γ is the antipodal quotient of $H(2D, 2)$;*
- *$c_i = (q^i - 1)/(q - 1)$ for $1 \leq i \leq D$, where q is an integer at least 2.*

In particular, $c_2 \geq 2$ and the girth of Γ is 4.

Case $D \geq 6$, bipartite

Theorem (Miklavic)

Assume Γ is bipartite with $D \geq 6$. Then $c_2 \geq 2$ (and therefore the girth of Γ is 4).

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Assume Γ is bipartite with $D \geq 6$. Then $c_2 \geq 2$ (and therefore the girth of Γ is 4).

The proofs of the above two theorems use the Terwilliger algebra of a Q-polynomial distance-regular graph. The proofs don't work for the cases $D = 4$ and $D = 5$.

Cases $D = 4$ and $D = 5$ - an alternative approach

Theorem (Miklavic)

Assume Γ is bipartite with $c_2 = 1$. Then

- $c_{i+1} - 1$ *divides* $c_i(c_i - 1)$ *for* $2 \leq i \leq D - 1$*;*
- $b_{i-1} - 1$ *divides* $b_i(b_i - 1)$ *for* $1 \leq i \leq D - 1$*.*

Moreover, $b_{D-1} \geq 2$*.*

Case $D = 4$ - an alternative approach

Theorem (Miklavic 2007)

There is no Q-polynomial bipartite distance-regular graph with $D = 4$ and $c_2 = 1$.

Case $D = 5$ - an alternative approach

Step one: find all integers $b_0 > b_3 \geq b_4$ such that

- $b_0 - 1$ divides $b_3(b_3 - 1)$;
- b_3 divides $b_4(b_4 - 1)$;
- $b_0 - b_4 - 1$ divides $(b_0 - b_3)(b_0 - b_3 - 1)$;
- $b_0 - 1$ divides $(b_0 - b_4)(b_0 - b_4 - 1)$

Cases $D = 5$ - an alternative approach

Step two: show that there is no bipartite Q-polynomial graph with girth 6 and with the above intersection numbers.

Cases $D = 5$ - an alternative approach

I am almost sure that I can prove that $b_4 \neq b_3 - 1$.

Cases $D = 5$ - an alternative approach

Up to 10000 there is only one example: $b_0 = 7568$, $b_3 = 6111$, $b_4 = 3290$.