Isolating highly connected induced subgraphs

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January 8, 2018

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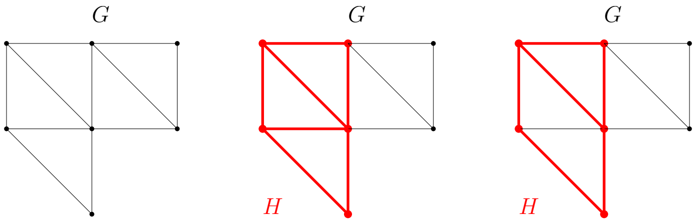
Definition

An *induced subgraph* of a graph G is any graph H s.t. $V(H) \subseteq V(G)$ and for all distinct $u, v \in V(H)$, $uv \in E(H)$ iff $uv \in E(G)$.



an induced subgraph

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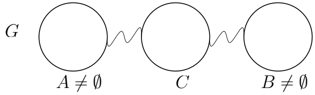

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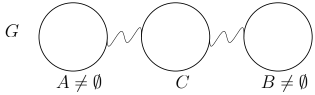

Definition

Let $k \in \mathbb{N}^+$. A graph is *k-connected* if it has $\geq k+1$ vertices and does not admit a cutset of size $\leq k-1$.

Theorem [Mader, 1972]

Let $k \in \mathbb{N}^+$, and let G be a graph. If $d(G) \geq 4k$,[a] then G contains a $(k+1)$-connected induced subgraph.

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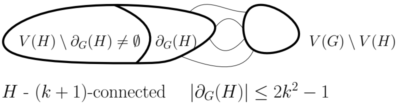

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$\forall d \in \mathbb{N}^+$, there is a graph of average degree $\geq d$, all of whose 2-connected induced subgraphs have frontier of size $\geq d$.

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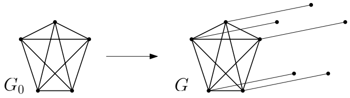

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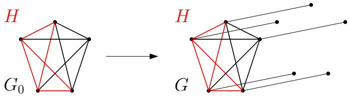

Let H be a 2-connected induced subgraph of G. Then H is an induced subgraph of G_0; because of the pendant edges, $\partial_G(H) = V(H)$. Furthermore, H contains a cycle, and so $|V(H)| \geq \mathrm{girth}(G) = d$, and consequently, $|\partial_G(H)| \geq d$. Q.E.D.

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Proof:

G is k^2-regular.
$\implies \delta(G) = k^2$

Let H be a $(k+1)$-connected induced subgraph of G.

Then H lies entirely inside one copy of G_0, and $\delta(H) \geq k+1$.

$\implies u_1, \ldots, u_k \notin V(H)$

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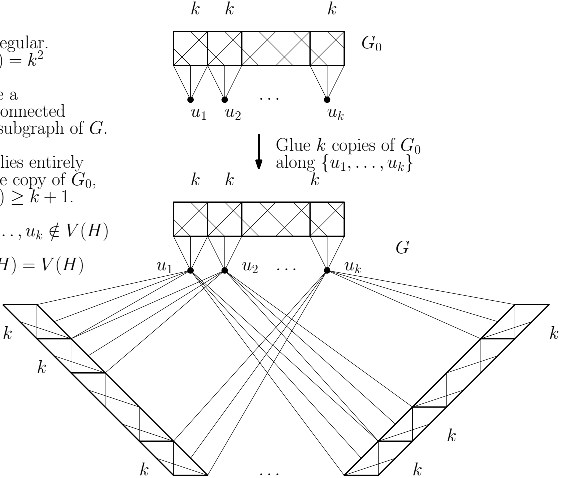

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Let $k \in \mathbb{N}^+$, and let G be a graph. Then at least one of the following holds:

(a) G is $(k+1)$-connected;

(b) G admits a cut-partition (A, B, C) s.t. $G[A \cup C]$ is $(k+1)$-connected and $|C| \leq 2k^2 - 1$;

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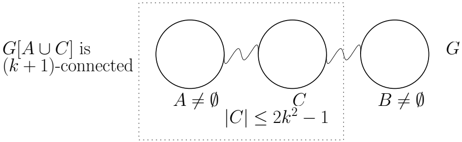

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Theorem 1' $\Rightarrow$ Theorem 1:

If (a) holds:

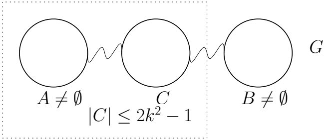
$H := G$

$\partial_G(H) = \emptyset$

If (b) holds:

$H := G[A \cup C]$ is $(k+1)$-connected

$\partial_G(H) \subseteq C$



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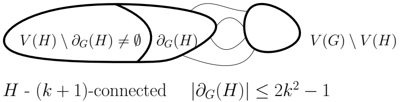
Theorem 1 $\Rightarrow$ Theorem 1':

Assume (c) is false.

$\implies \delta_G(H) > 2k^2 - 1$



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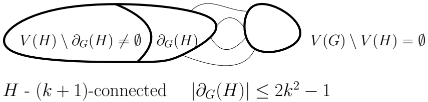
$\Longrightarrow G = H$



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Case 2: $V(G) \setminus V(H) \neq \emptyset$

$A := V(H) \setminus \partial_G(H)$
$B := V(G) \setminus V(H)$
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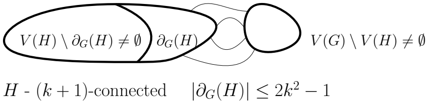

Theorem [Alon, Kleitman, Saks, Seymour, Thomassen, 1987]

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Proof (using Theorem 1'): Let G be s.t. $\chi(G) > c + 2k^2 - 1$. We must exhibit a $(k+1)$-connected induced subgraph H of G s.t. $\chi(H) > c$.

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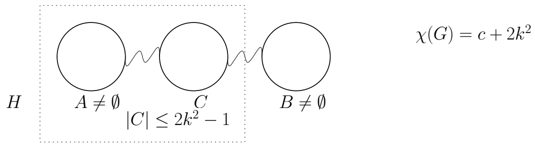
We may assume that G is not $(k+1)$-connected (otherwise, we set $H := G$, and we are done). Thus, (a) from Theorem 1' is false.

Proof (cont.): Thus, (b) from Theorem 1' holds. Let (A, B, C) be as in (b) from Theorem 1', and set $H := G[A \cup C]$. Then H is $(k+1)$-connected; we must show that $\chi(H) > c$.



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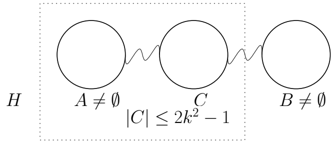

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$\chi(G) = c + 2k^2$

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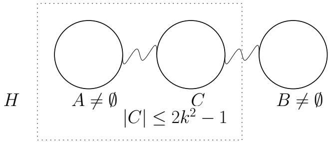
Since G is vertex-critical, $\chi(\underbrace{G \setminus A}_{=G[B \cup C]}) \leq \chi(G) - 1 = c + 2k^2 - 1$.



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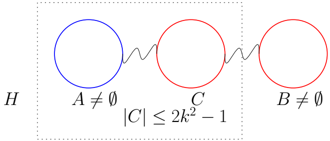

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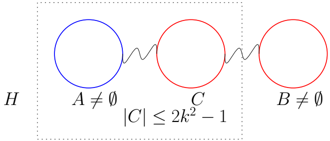

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We properly color $G \setminus A = G[B \cup C]$ with $c + 2k^2 - 1$ colors.

At most $|C| \leq 2k^2 - 1$ of those colors are used on C; consequently, at least c of our $c + 2k^2 - 1$ colors remain "unused" on C.

Use these c "unused" colors to properly color $G[A]$.

We now have a proper coloring of G that uses only $c + 2k^2 - 1$ colors, contrary to the fact that $\chi(G) = c + 2k^2$. Q.E.D.

Corollary [P., Thomassé, Trotignon, 2016]

Let $k, c \in \mathbb{N}^+$. Then every graph of chromatic number greater than $\underline{c + 2k^2 - 1}$ has a $(k+1)$-connected induced subgraph of chromatic number greater than c.

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Theorem 2 [P., Thomassé, Trotignon, 2016]

Let $k, c \in \mathbb{N}^+$. Then every graph of chromatic number greater than $\underline{\max\{c + 2k - 2, 2k^2\}}$ has a $(k+1)$-connected induced subgraph of chromatic number greater than c.

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- Theorem 2 does **not** follow from Theorem 1' (equivalently: Theorem 1). It can, however, be derived form a lemma (Lemma 1) that we used to prove Theorem 1'.

Definition

Let $k \in \mathbb{N}^+$, and let G be a graph.

1. for all $v \in V(G)$ and $Z \subseteq V(G) \setminus \{v\}$,[a]

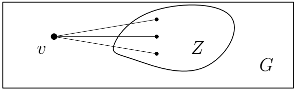
$$w_Z(v) = \begin{cases} 1 & \text{if } d_Z(v) = 0 \\ d_Z(v) & \text{if } 1 \leq d_Z(v) \leq k \\ k & \text{if } d_Z(v) \geq k+1 \end{cases}$$

2. for all disjoint sets $Y, Z \subseteq V(G)$, $w_Z(Y) = \sum_{v \in Y} w_Z(v)$.[b]

---

[a] $d_Z(v)$ = number of neighbors that v has in Z

[b] $\Longrightarrow |Y| \leq w_Z(Y) \leq k|Y|$



$d_Z(v) = |N_G(v) \cap Z|$

Lemma 1 [P., Thomassé, Trotignon, 2016]

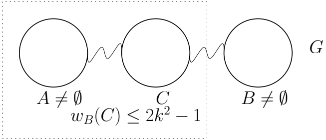
Let $k \in \mathbb{N}^+$, and let G be a graph. Then at least one of the following holds:

(a) G is $(k+1)$-connected;

(b) G admits a cut-partition (A, B, C) s.t. $G[A \cup C]$ is $(k+1)$-connected and $\underline{w_B(C) \leq 2k^2 - 1}$;[a]

(c) G contains a vertex of degree at most $2k^2 - 1$.

[a] Consequently, $|C| \leq w_B(C) \leq 2k^2 - 1$.



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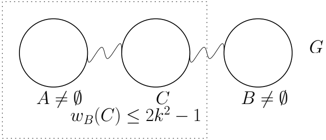

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- Clearly, Lemma 1 implies Theorem 1'.

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[a] Consequently, $|C| \leq w_B(C) \leq 2k^2 - 1$.

Proof: We assume that (a) and (c) are false (i.e. G is not $(k+1)$-connected, and $\delta(G) \geq 2k^2$), and we prove (b).

Claim 1: G admits a cut-partition (A, B, C) *s.t.* $w_B(C) \leq 2k^2 - 1$.

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Proof of Claim 1: Since G is **not** $(k+1)$-connected, either

(1) $|V(G)| \leq k + 1$, or

(2) G admits a cutset of size $\leq k$.

However,

$$|V(G)| \geq \delta(G) + 1 \geq 2k^2 + 1 \geq k + 2.$$

and so (1) is false. Thus, (2) is true.

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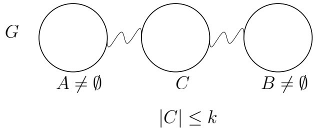
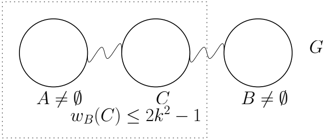


Then $w_B(C) \leq k|C| \leq k^2 \leq 2k^2 - 1$. This proves Claim 1.

Proof (cont.): Let (A, B, C) be a cut-partition of G with $w_B(C) \leq 2k^2 - 1$, and subject to that, chosen so that $A \cup C$ is minimal.[4]



[4] Thus, there does **not** exist a cut-partition (A', B', C') of G s.t. $w_{B'}(C') \leq 2k^2 - 1$ and $A' \cup C' \subsetneq A \cup C$.

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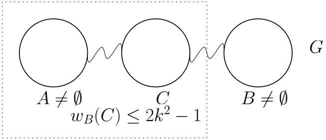

We must show that $G[A \cup C]$ is $(k+1)$-connected, that is, that

- $|A \cup C| \geq k + 2$, and
- $G[A \cup C]$ does not admit a cutset of size $\leq k$.

This will imply that (A, B, C) satisfies (b).

[4] Thus, there does **not** exist a cut-partition (A', B', C') of G s.t. $w_{B'}(C') \leq 2k^2 - 1$ and $A' \cup C' \subsetneq A \cup C$.

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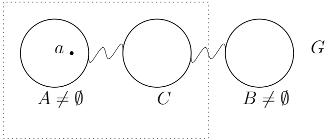
Claim 2: $|A \cup C| \geq k + 2$.

Proof of Claim 2: Suppose otherwise, i.e. $|A \cup C| \leq k + 1$.



Fix $a \in A$. Then

$$\deg_G(a) \leq |A \cup C| - 1 \leq k < 2k^2 \leq \delta(G),$$

a contradiction. This proves Claim 2.

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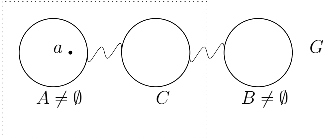

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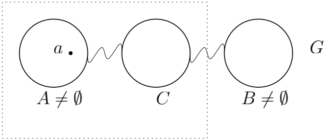

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$$\deg_G(a) \leq |A \cup C| - 1 \leq k < 2k^2 \leq \delta(G),$$

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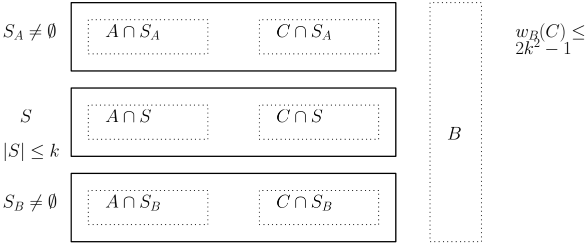
- It remains to show that $G[A \cup C]$ does not admit a cutset of size $\leq k$.
- Suppose otherwise, i.e. $G[A \cup C]$ admits a cutset of size $\leq k$.

Proof (cont.): Let (S_A, S_B, S) be a cut-partition of $G[A \cup C]$ with $|S| \leq k$.



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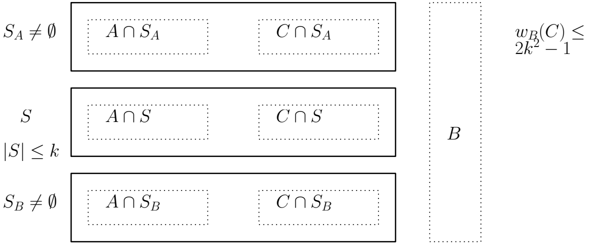
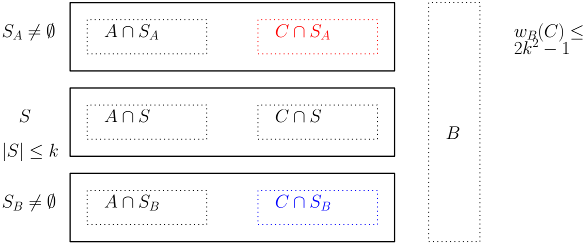


Goal: Derive a contradiction by either

- exhibiting a vertex $v \in V(G)$ s.t. $\deg_G(v) \leq 2k^2 - 1$ (contrary to the fact that $\delta(G) \geq 2k^2$), or
- exhibiting a cut-partition (A', B', C') of G s.t. $w_{B'}(C') \leq 2k^2 - 1$ and $A' \cup C' \subsetneq A \cup C$ (contrary to the minimality of $A \cup C$).


$S_A \neq \emptyset$
$A \cap S_A$
$C \cap S_A$
S
$|S| \leq k$
$A \cap S$
$C \cap S$
$S_B \neq \emptyset$
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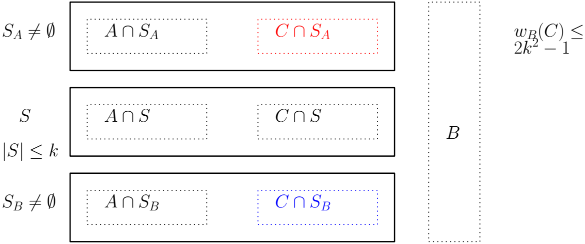
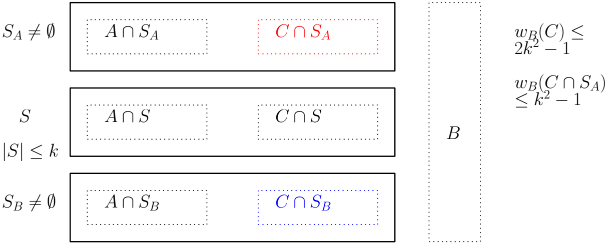


Clearly, $w_B(C \cap S_A) + w_B(C \cap S_B) \leq w_B(C) \leq 2k^2 - 1$.

$\Longrightarrow$ Either $w_B(C \cap S_A) \leq k^2 - 1$ or $w_B(C \cap S_B) \leq k^2 - 1$.



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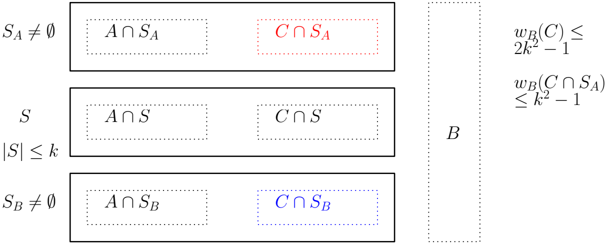
By symmetry, we may assume that $w_B(C \cap S_A) \leq k^2 - 1$.



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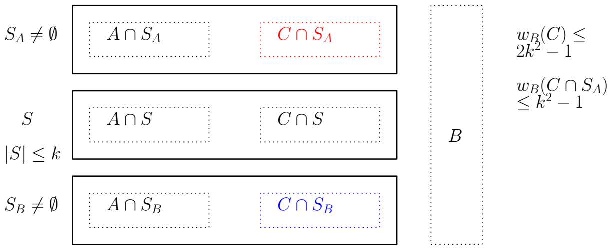
Claim 3: $A \cap S_A = \emptyset$.



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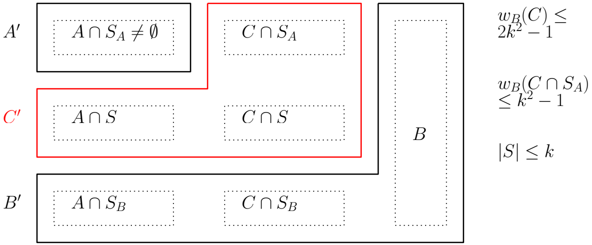
By symmetry, we may assume that $w_B(C \cap S_A) \leq k^2 - 1$.

Claim 3: $A \cap S_A = \emptyset$.

Proof of Claim 3: Suppose otherwise, i.e. $A \cap S_A \neq \emptyset$.

Proof (cont.): Proof of Claim 3 (cont.):

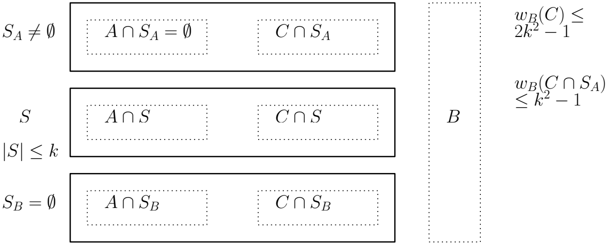


(A', B', C') is a cut-partition of G with $A' \cup C' \subsetneq A \cup C$, and

$$\begin{aligned} w_{B'}(C') &= w_{B'}(S) + w_{B'}(C \cap S_A) \\ &\leq k|S| + w_B(C \cap S_A) \\ &\leq k^2 + (k^2 - 1) \leq 2k^2 - 1, \end{aligned}$$

a contradiction to the minimality of $A \cup C$. This proves Claim 3 (i.e. $A \cap S_A = \emptyset$).

Proof (cont.): Since $S_A \neq \emptyset$, it follows that $C \cap S_A \neq \emptyset$.



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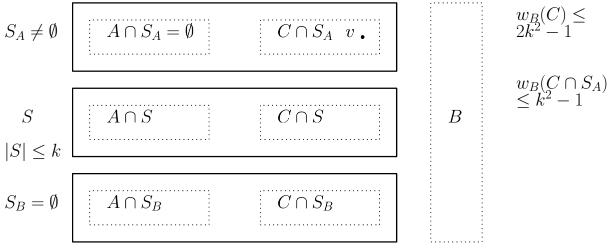

Claim 4: For all $v \in C \cap S_A$, $w_B(v) = k$. Consequently, $w_B(C \cap S_A) = k|C \cap S_A|$.

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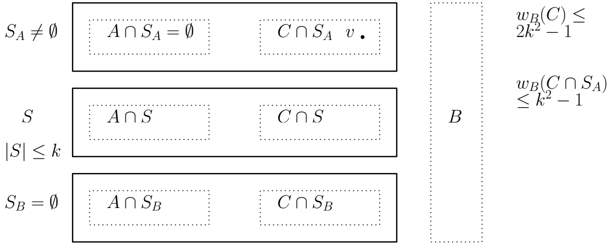

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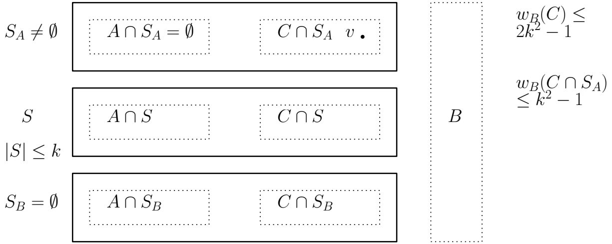
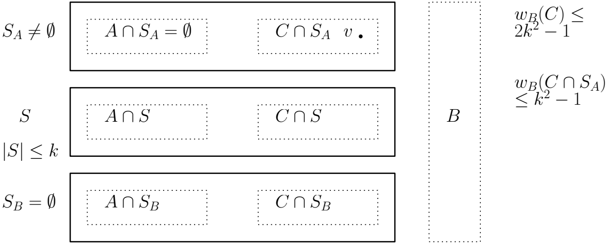
Proof of Claim 4: Fix $v \in C \cap S_A$. By the definition of $w_B(v)$, it suffices to show that $d_B(v) > w_B(v)$.

Proof (cont.): Proof of Claim 4 (cont.): Recall: We need to show that $d_B(v) > w_B(v)$.



$$\begin{aligned} 2k^2 \leq \delta(G) \leq \deg_G(v) &\leq |(C \cap S_A) \setminus \{v\}| + |S| + d_B(v) \\ &\leq w_B((C \cap S_A) \setminus \{v\}) + |S| + d_B(v) \\ &\leq w_B(C \cap S_A) - w_B(v) + |S| + d_B(v) \\ &\leq (k^2 - 1) - w_B(v) + k + d_B(v). \end{aligned}$$

Proof (cont.): Proof of Claim 4 (cont.): Recall: We need to show that $d_B(v) > w_B(v)$.



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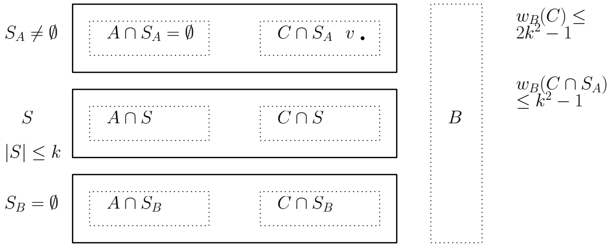
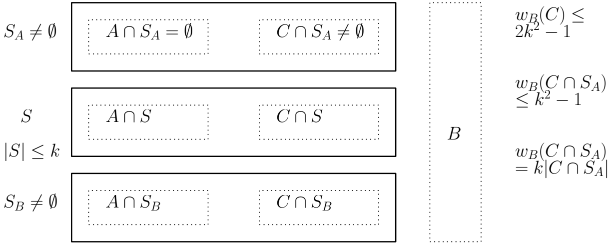


$$\implies 2k^2 \leq (k^2 - 1) - w_B(v) + k + d_B(v)$$

$$\implies d_B(v) \geq w_B(v) + k^2 - k + 1 > w_B(v)$$

This proves Claim 4 (in particular, $w_B(C \cap S_A) = k|C \cap S_A|$).

Proof (cont.):



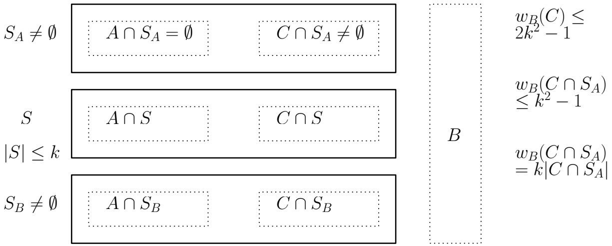
Claim 5: $A \cap S_B \neq \emptyset$.

Proof of Claim 5:

$$\begin{aligned} |C \setminus S_A| &\leq w_B(C \setminus S_A) \\ &\leq w_B(C) - w_B(C \cap S_A) \\ &\leq (2k^2 - 1) - k|C \cap S_A| \end{aligned}$$

$$\begin{aligned} |C| &\leq |C \setminus S_A| + |C \cap S_A| \\ &\leq (2k^2 - 1) - (k - 1)|C \cap S_A| \end{aligned}$$

Proof (cont.): Proof of Claim 5 (cont.): Recall that $|C| \leq (2k^2 - 1) - (k-1)|C \cap S_A|$.



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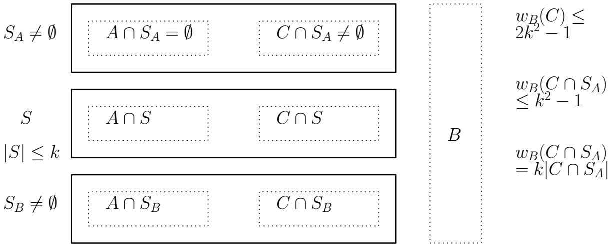
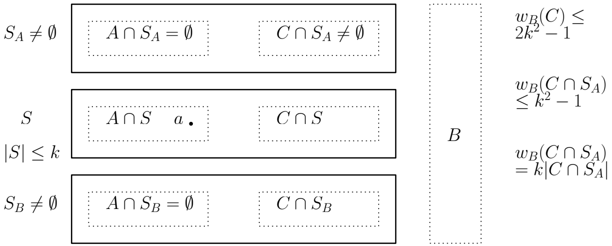


Proof (cont.): Proof of Claim 5 (cont.): Recall that $|C| \leq (2k^2 - 1) - (k-1)|C \cap S_A|$.

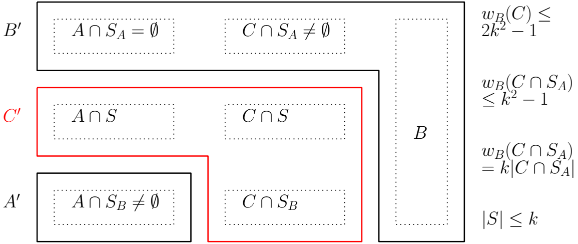
Recall: We need to show that $A \cap S_B \neq \emptyset$. Suppose otherwise, i.e. $A \cap S_B = \emptyset$. Fix $a \in A$ ($\Longrightarrow a \in A \cap S$).



$$
\begin{aligned}
\deg_G(a) &\leq |(S \cup C) \setminus \{a\}| \leq |S \setminus \{a\}| + |C| \\
&\leq (k-1) + (2k^2 - 1) - (k-1)|C \cap S_A| \\
&= (2k^2 - 1) - (k-1)(|C \cap S_A| - 1) \\
&\leq 2k^2 - 1 < \delta(G),
\end{aligned}
$$

a contradiction. This proves Claim 5 (i.e. $A \cap S_B \neq \emptyset$).

Proof (cont.):



B'
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C'
$A \cap S$
$C \cap S$
B
A'
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$C \cap S_B$
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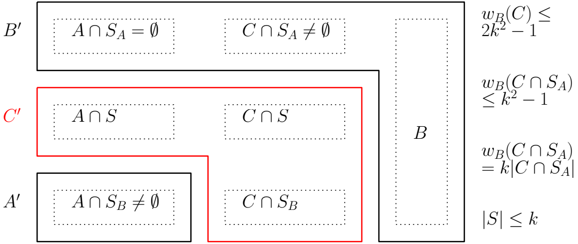
$w_B(C \cap S_A) = k|C \cap S_A|$
$|S| \leq k$

Proof (cont.):



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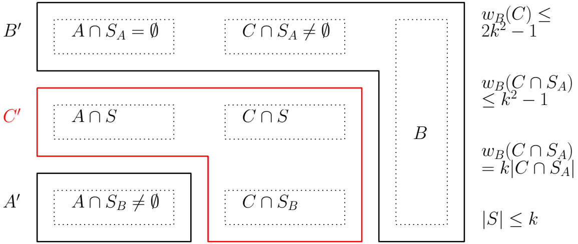

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For this, we need to show that:

1. $A' \cup C' \subsetneq A \cup C$;
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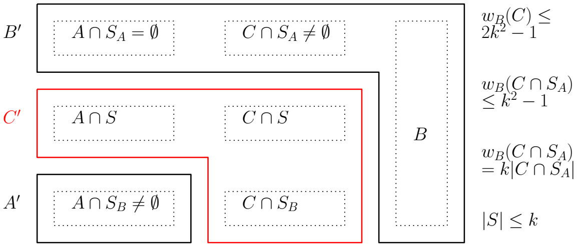
Proof (cont.):



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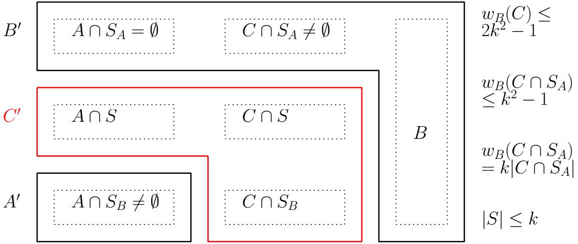
Proof (cont.):

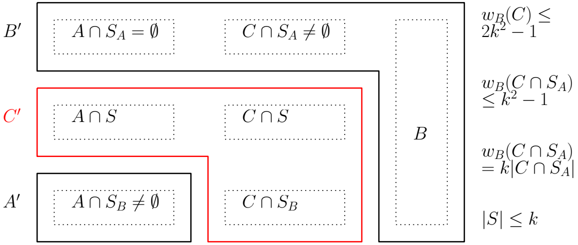


Our goal is to show that (A', B', C') contradicts the choice of (A, B, C).

For this, we need to show that:

1. $A' \cup C' \subsetneq A \cup C$;
   - This follows from the fact that $C \cap S_A \neq \emptyset$
2. $w_{B'}(C') \leq 2k^2 - 1$.
   - Since $w_B(C) \leq 2k^2 - 1$, it suffices to show that $w_{B'}(C') \leq w_B(C)$.

Proof (cont.):



Claim 6: $w_{B'}(C') \leq w_B(C)$.

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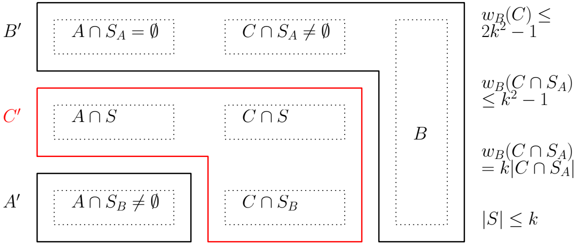


Claim 6: $w_{B'}(C') \leq w_B(C)$.

Proof of Claim 6: When we "move" from $w_B(C)$ to $w_{B'}(C')$:

- we "lose" $w_B(C \cap S_A) = k|C \cap S_A|$, and
- we "gain" $\leq w_{S_A}(S) = w_{C \cap S_A}(S) \leq |S||C \cap S_A| \leq k|C \cap S_A|$.

Thus, $w_{B'}(C') \leq w_B(C)$. This proves Claim 6. Q.E.D.

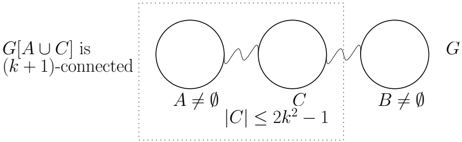
Theorem 1' [P., Thomassé, Trotignon, 2016]

Let $k \in \mathbb{N}^+$, and let G be a graph. Then at least one of the following holds:

(a) G is $(k+1)$-connected;

(b) G admits a cut-partition (A, B, C) s.t. $G[A \cup C]$ is $(k+1)$-connected and $|C| \leq 2k^2 - 1$;

(c) G contains a vertex of degree at most $2k^2 - 1$.



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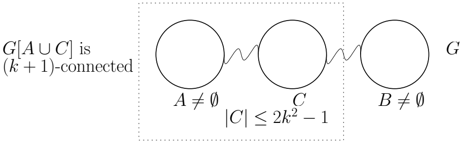

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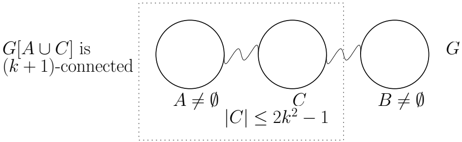

- For $k = 2$, the optimal bound is 5 (rather than $2k^2 - 1 = 7$).
  - The proof is completely different from that of Theorem 1', and it does not (seem to) generalize to higher values of k.

Theorem [Alon, Kleitman, Saks, Seymour, Thomassen, 1987]

Let $k, c \in \mathbb{N}^+$. Then every graph of chromatic number greater than $\underline{\max\{c + 10k^2 + 1, 100k^3\}}$ has a $(k+1)$-connected induced subgraph of chromatic number greater than c.

Theorem [Chudnovsky, P., Scott, Trotignon, 2013]

Let $k, c \in \mathbb{N}^+$. Then every graph of chromatic number greater than $\underline{\max\{c + 2k^2, 2k^2 + k\}}$ has a $(k+1)$-connected induced subgraph of chromatic number greater than c.

Theorem 2 [P., Thomassé, Trotignon, 2016]

Let $k, c \in \mathbb{N}^+$. Then every graph of chromatic number greater than $\underline{\max\{c + 2k - 2, 2k^2\}}$ has a $(k+1)$-connected induced subgraph of chromatic number greater than c.

That's all.

Thanks for listening!

I. Penev, S. Thomassé, N. Tortignon, "Isolating highly connected induced subgraphs", *SIAM Journal on Discrete Mathematics*, 30(1) (2016), 592–619.

arXiv:1406.1671