Automorphism groups of non-edge transitive Rose Windows graphs

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Rose window graphs

For an integer $n \geq 3$ and integers $1 \leq a, r \leq n-1$, $r \neq n/2$, the Rose Window graph $R_n(a, r)$ has vertex set $V = \{A_i, B_i \mid i \in \mathbb{Z}_n\}$ and four types of edges:

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- Rim edges $\{\{A_i, A_{i+1}\} \mid i \in \mathbb{Z}_n\}$;
- In-Spoke edges $\{\{A_i, B_i\} \mid i \in \mathbb{Z}_n\}$;
- Out-Spoke edges $\{\{B_i, A_{i+a}\} \mid i \in \mathbb{Z}_n\}$;
- Hub edges $\{\{B_i, B_{i+r}\} \mid i \in \mathbb{Z}_n\}$.

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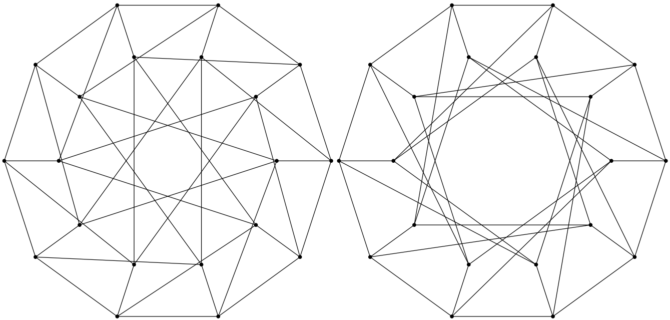
- Out-Spoke edges $\{\{B_i, A_{i+a}\} \mid i \in \mathbb{Z}_n\}$;
- Hub edges $\{\{B_i, B_{i+r}\} \mid i \in \mathbb{Z}_n\}$.

All arithmetics with a, r and vertex subscripts is assumed to be done in $\mathbb{Z}_n$. Note that $R_n(a, r) = R_n(a, -r)$.

Rose window graphs



Rose window graphs

- Introduced by Steve Wilson in 2008
- Motivation: maps, generalization of $GPG(n, r)$

Rose window graphs - general problem

Given n, a, r, find the automorphism group of $R_n(a, r)$.

Rose window graphs - automorphism group

Let G be the automorphism group of $R_n(a,r)$. Define $\rho : V \to V$ and $\mu : V \to V$ by

$$\rho(A_i) = A_{i+1} \quad \text{and} \quad \rho(B_i) = B_{i+1} \qquad (i \in \mathbb{Z}_n),$$
$$\mu(A_i) = A_{-i} \quad \text{and} \quad \mu(B_i) = B_{-a-i} \qquad (i \in \mathbb{Z}_n).$$

Rose window graphs - automorphism group

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Note that $\rho, \mu \in G$, and therefore $\langle \rho, \mu \rangle \le G$. The action of $\langle \rho, \mu \rangle$ on the set of edges of $R_n(a,r)$ has three orbits: the set of rim edges, the set of hub edges and the set of spoke edges.

Rose window graphs - automorphism group

Lemma

Let $R_n(a, r)$ denote a Rose Window graph. Then the following (i)–(iii) *are equivalent:*

(i) *$R_n(a, r)$ is edge-transitive.*

(ii) *There is an automorphism of $R_n(a, r)$ which sends a rim edge to a spoke edge.*

(iii) *There is an automorphism of $R_n(a, r)$ which sends a spoke edge to a hub edge.*

Rose window graphs - automorphism group

Theorem

Let $R_n(a, r)$ denote a Rose Window graph and let G be its group of automorphisms. Then there exists $\sigma \in G$ sending rim edges to hub edges and vice-versa if and only if one of the following (i), (ii) *holds:*

(i) $r^2 \equiv \pm 1 \pmod{n}$ *and* $ra \equiv ta \pmod{n}$, *where* $t \in \{-1, 1\}$;

(ii) n *is divisible by* 4, $a = n/2$ *and* $(r^2 + n/2) \equiv \pm 1 \pmod{n}$.

Rose window graphs - automorphism group

Let $N = gcd(n, r)$ denote the number of "inner" cycles of $R_n(a, r)$, and let $L = n/N$ denotes the length of these inner cycles. Assume for a moment that n is even. For $0 \leq \ell \leq n/2 - 1$ let

$$\alpha_\ell = (B_\ell, B_{\ell+n/2}).$$

Rose window graphs - automorphism group

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$$\alpha_\ell = (B_\ell, B_{\ell+n/2}).$$

For $0 \leq \ell \leq N - 1$ let

$$\beta_\ell = (B_\ell, B_{\ell+n/2})(B_{\ell+N}, B_{\ell+N+n/2}) \cdots (B_{\ell+n/2-N}, B_{\ell+n-N}).$$

Rose window graphs - automorphism group

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For $0 \leq \ell \leq N/2 - 1$ let

$$\gamma_\ell = (B_\ell, B_{\ell+n/2})(B_{\ell+N}, B_{\ell+N+n/2}) \cdots (B_{\ell+n-N}, B_{\ell+n-N+n/2}).$$

Rose window graphs - automorphism group

Lemma

Let $R_n(a, r)$ denote a Rose Window graph and let G be its group of automorphisms. Assume n is even. Then the following (i)-(iii) hold.

(i) $\alpha_\ell = \rho^\ell \alpha_0 \rho^{-\ell}$ *for* $0 \leq \ell \leq n/2 - 1$.

(ii) $\beta_\ell = \rho^\ell \beta_0 \rho^{-\ell}$ *for* $0 \leq \ell \leq N - 1$.

(iii) $\gamma_\ell = \rho^\ell \gamma_0 \rho^{-\ell}$ *for* $0 \leq \ell \leq N/2 - 1$.

Rose window graphs - automorphism group

Lemma

Let $R_n(a, r)$ denote a Rose Window graph. Assume n is even and $a = n/2$. Then the following (i)-(iii) hold.

(i) *If $L = 4$ then α_ℓ is an automorphism or $R_n(n/2, r)$ for $0 \leq \ell \leq n/2 - 1$.*

(ii) *If L is even, then β_ℓ is an automorphism or $R_n(n/2, r)$ for $0 \leq \ell \leq N - 1$.*

(iii) *If L is odd then γ_ℓ is an automorphism or $R_n(n/2, r)$ for $0 \leq \ell \leq N/2 - 1$.*

Rose window graphs - automorphism group

Lemma

Let $R_n(a, r)$ denote a Rose Window graph and let G be its group of automorphisms. Let $G_{\mathcal{A}}$ be the point-wise stabiliser of $\{A_0, A_1, \ldots, A_{n-1}\}$ in G. Then the following (i)–(iv) hold.

(i) *If $a \neq n/2$ then $G_{\mathcal{A}}$ is trivial.*

(ii) *If $a = n/2$ and $L = 4$, then $G_{\mathcal{A}} = \langle \alpha_0, \alpha_1, \ldots \alpha_{n/2-1} \rangle$.*

(iii) *If $a = n/2$, L is even and $L \neq 4$, then $G_{\mathcal{A}} = \langle \beta_0, \beta_1, \ldots \beta_{N-1} \rangle$.*

(iv) *If $a = n/2$ and L is odd, then $G_{\mathcal{A}} = \langle \gamma_0, \gamma_1, \ldots \gamma_{N/2-1} \rangle$.*

Rose window graphs - automorphism group

Lemma

Let $R_n(a, r)$ denote a Rose Window graph and let G be its group of automorphisms. Let $G_{\{\mathcal{A}\}}$ be the set-wise stabiliser of $\{A_0, A_1, \ldots, A_{n-1}\}$ in G. Then the following (i)–(iv) hold.

(i) *If $a \neq n/2$ then $G_{\{\mathcal{A}\}} = \langle \rho, \mu \rangle$.*

(ii) *If $a = n/2$ and $L = 4$, then $G_{\{\mathcal{A}\}} = \langle \rho, \mu, \alpha_0 \rangle$.*

(iii) *If $a = n/2$, L is even and $L \neq 4$, then $G_{\{\mathcal{A}\}} = \langle \rho, \mu, \beta_0 \rangle$.*

(iv) *If $a = n/2$ and L is odd, then $G_{\{\mathcal{A}\}} = \langle \rho, \mu, \gamma_0 \rangle$.*

Rose window graphs - automorphism group

Corollary

Let $R_n(a,r)$ denote a Rose Window graph and let G be its group of automorphisms. Assume G has three orbits on edge-set of $R_n(a,r)$ (that is, $R_n(a,r)$ does not satisfy non of the conditions (i) and (ii) of Theorem 2). Then the following (i)–(iv) *hold.*

(i) *If $a \neq n/2$ then $G = \langle \rho, \mu \rangle$.*

(ii) *If $a = n/2$ and $L = 4$, then $G = \langle \rho, \mu, \alpha_0 \rangle$.*

(iii) *If $a = n/2$, L is even and $L \neq 4$, then $G = \langle \rho, \mu, \beta_0 \rangle$.*

(iv) *If $a = n/2$ and L is odd, then $G = \langle \rho, \mu, \gamma_0 \rangle$.*

Rose window graphs - automorphism group

Theorem

Let $R_n(a,r)$ denote a Rose Window graph and let G be its group of automorphisms. Assume $a \neq n/2$, $r^2 \equiv \pm 1 \pmod{n}$ and $ra \equiv -a \pmod{n}$. Then $G = \langle \rho, \mu, \delta \rangle$, where δ is defined by $\delta(A_i) = B_{ri}$ and $\delta(B_i) = A_{ri}$.

Rose window graphs - automorphism group

Theorem

Let $R_n(a, r)$ denote a Rose Window graph and let G be its group of automorphisms. Assume $a = n/2$, $r^2 \equiv \pm 1 \pmod{n}$ and $ra \equiv -a \pmod{n}$. Then $G = \langle \rho, \mu, \beta_0, \delta \rangle$.

Rose window graphs - automorphism group

Theorem

Assume n is divisible by 4, r is odd, $a = n/2$ and $(r^2 + n/2) \equiv \pm 1 \pmod{n}$. Then $G = \langle \rho, \mu, \beta_0, \gamma \rangle$, here γ is defined by $\gamma(A_i) = B_{ri}$ and $\gamma(B_i) = A_{(r+n/2)i}$.