Univerza v Ljubljani
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Geometric interpolation by parametric polynomial curves

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Outline

1. **Motivation**
2. **General conjecture**
3. **Planar case**
4. **Nonasymptotic analysis**
5. **Special curves**
6. **Open problems**

Motivation

- Standard problem in CAGD:
  - Points $\boldsymbol{T}_j \in \mathbb{R}^d$ ($d \geq 2$), $j = 0, 1, \ldots, k$, are given.
  - Find parametric polynomial $\boldsymbol{p}$ such that

$$\boldsymbol{p}(t_j) = \boldsymbol{T}_j, \quad j = 0, 1, \ldots, k.$$

  - If k is large, replace $\boldsymbol{p}$ by spline $\boldsymbol{s}$.

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$$\boldsymbol{p}(t_j) = \boldsymbol{T}_j, \quad j = 0, 1, \ldots, k.$$

    - If k is large, replace $\boldsymbol{p}$ by spline $\boldsymbol{s}$.
- If the sequence $\{t_j\}_{j=0}^{k}$ is known, construction of $\boldsymbol{p}$ is a linear problem.
- Different choices of $\{t_j\}_{j=0}^{k}$ give different curves.
- Degree of $\boldsymbol{p}$ is k in general.

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Possible choices for $\{t_j\}_{j=1}^{k}$ are:

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- Chordal:

$$t_{j+1} = t_j + \|\Delta \boldsymbol{T}_j\| = \sum_{\ell=0}^{j} \|\Delta \boldsymbol{T}_\ell\|, \quad \Delta \boldsymbol{T}_\ell = \boldsymbol{T}_{\ell+1} - \boldsymbol{T}_\ell.$$

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- Lee's generalization:

$$t_{j+1} = t_j + \|\Delta \boldsymbol{T}_j\|^\alpha = \sum_{\ell=0}^{n} \|\Delta \boldsymbol{T}_\ell\|^\alpha, \quad j = 0, 1, \dots, k-1,$$

where $\alpha \in [0, 1]$. The most known one is centripetal ($\alpha = 1/2$).

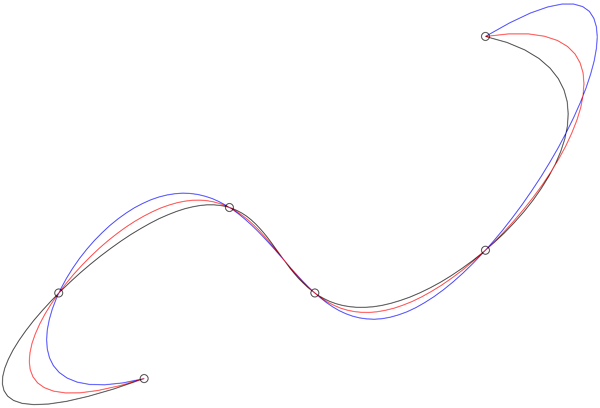

Figure : Various parameterizations by quintic polynomial: uniform (black), chordal (blue) and centripetal (red).

- Number of points interpolated by polynomial curve of degree $\leq k$ is at most $k + 1$.
- Expected approximation order in case of “dense” data is $k + 1$.
- Unique solution always exists.
- Expected computational time is $\mathcal{O}(d\,k^2)$.

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- Unique solution always exists.
- Expected computational time is $\mathcal{O}(d\,k^2)$.
- Is it possible to increase the number of interpolated points by polynomial curve of the same degree k?

How many points can be interpolated by planar parametric parabola?

- 3?
- 4?
- 5 or more?

How many points can be interpolated by planar parametric parabola?

- 3?. . . always
- 4?. . . sometimes
- 5 or more?. . . pure luck

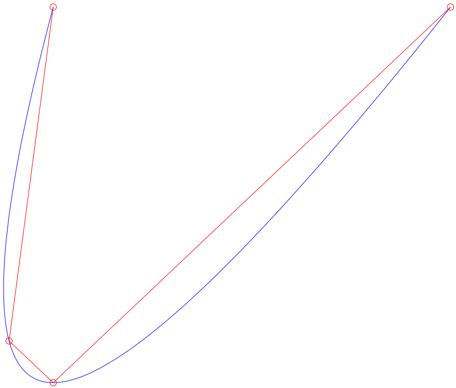

Figure : Four points interpolated by a parametric parabola.

Detailed analysis: K. Mørken, Parametric interpolation by quadratic polynomials in the plane.

Conjecture (Höllig and Koch(1996))

Parametric polynomial curve of degree k in $\mathbb{R}^d$ can, in general, interpolate

$$\left\lfloor \frac{d\,(k+1)-2}{d-1} \right\rfloor$$

points.

If the conjecture holds true, the approximation order of interpolating polynomial might be much higher than in the functional case.

Maybe even a shape of the resulting curve is satisfactory?!

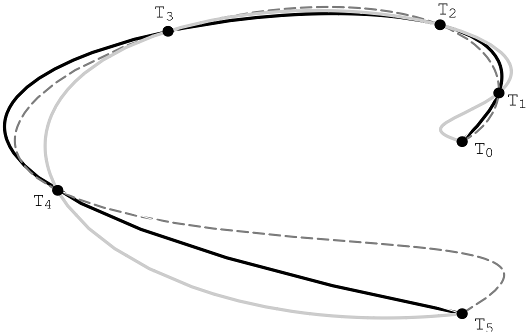

Figure : Cubic geometric interpolant on 6 points (solid), quintic chordal parameterization (doted), quintic uniform parameterization (gray).

- Probably the first serious attempt to analyze geometric (cubic) interpolant goes back to 1987:
  C. de Boor, K. Höllig, and M. Sabin: High accuracy geometric Hermite interpolation. Comput. Aided Geom. Design 4 (1987), no. 4, 269–278.

- Probably the first serious attempt to analyze geometric (cubic) interpolant goes back to 1987:
  C. de Boor, K. Höllig, and M. Sabin: High accuracy geometric Hermite interpolation. Comput. Aided Geom. Design 4 (1987), no. 4, 269–278.
- Asymptotic analysis of geometric Hermite interpolation of values, tangent directions and curvatures at two boundary points by planar cubic polynomial curve.
- Approximation order is 6, but there might be no solution.

- The most interesting case of the conjecture is $d = 2$.
- Nonasymptotic analysis is terribly complicated in general.
- Conjecture is still an open problem.
- Only a few generalizations to spline cases are known.
- It seems it is more or less theoretical issue.

Nonlinear equations in the planar case

- Equations:

$$\boldsymbol{p}_n(t_\ell) = \boldsymbol{T}_\ell, \quad \ell = 0, 1, \ldots, 2n-1.$$

- Unknowns t_ℓ are ordered as

$$t_0 < t_1 < \cdots < t_{2n-1}.$$

- We may assume $t_0 := 0$, $t_{2n-1} := 1$ (linear reparameterization).

- $\boldsymbol{t} := (t_\ell)_{\ell=1}^{2n-2}$ are not the only unknowns.
- Also the coefficients of the polynomial $\boldsymbol{p}_n$ have to be determined.

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- Also the coefficients of the polynomial $\boldsymbol{p}_n$ have to be determined.
- First part of the problem is nonlinear (hard).
- Second part is linear (easy).
- The problem can be split into two parts: finding $\boldsymbol{t}$ first and then the coefficients of $\boldsymbol{p}_n$.

Divided differences

- The equations for the unknown parameters $\boldsymbol{t}$ can be derived using linearly independent functionals (divided differences).
- One way is to choose

$$[t_0, t_1, \ldots, t_{n+j}], \quad j = 1, 2, \ldots, n-1.$$

- Applying $[t_0, t_1, \ldots, t_{n+j}]$ to the equations

$$\boldsymbol{p}_n(t_\ell) = \boldsymbol{T}_\ell$$

leads to

$$[t_0, t_1, \ldots, t_{n+j}]\boldsymbol{p}_n = \boldsymbol{0} = \sum_{\ell=0}^{n+j} \frac{\boldsymbol{T}_\ell}{\dot{\omega}_j(t_\ell)},$$

$$j = 1, \ldots, n-1,$$

where

$$\omega_j(t) := \prod_{\ell=0}^{n+j}(t - t_\ell), \quad \dot{\omega}_j(t) := \frac{d\omega_j(t)}{dt}.$$

- This gives $2n-2$ nonlinear equations for $2n-2$ unknowns $\boldsymbol{t} = (t_\ell)_{\ell=1}^{2\,n-2}$.
- Any sequence of $n+1$ parameters t_ℓ determine $\boldsymbol{p}_n$ uniquely.
- General analysis is unfortunately complicated $\longrightarrow$ asymptotic approach.

Asymptotic analysis

- Assumption: $\boldsymbol{T}_\ell$ are sampled from smooth convex planar curve
$$\boldsymbol{f}:[0,h]\to\mathbb{R}^2,$$
$\boldsymbol{f}(0)=(0,0)^T$, $\boldsymbol{f}'(0)=(1,0)^T$.

Asymptotic analysis

- Assumption: $\boldsymbol{T}_\ell$ are sampled from smooth convex planar curve

$$\boldsymbol{f} : [0, h] \to \mathbb{R}^2,$$

$$\boldsymbol{f}(0) = (0,0)^T,\ \boldsymbol{f}'(0) = (1,0)^T.$$

- The curve $\boldsymbol{f}$ is parametrized by the first component:

$$\boldsymbol{f}(x) = \begin{pmatrix} x \\ y(x) \end{pmatrix},$$

$$y(x) := \frac{1}{2} y''(0) x^2 + \mathcal{O}(x^3), \quad y''(0) > 0.$$

- Since h is small, the coordinate system should be scaled by the matrix

$$D_h = \operatorname{diag}\left(\frac{1}{h}, \frac{2}{h^2\, y''(0)}\right).$$

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- Suppose now

$$\eta_0 := 0 < \eta_1 < \cdots < \eta_{2n-2} < \eta_{2n-1} := 1,$$

are the (given) parameters, for which

$$\boldsymbol{T_\ell} = D_h \boldsymbol{f}(\eta_\ell h), \quad \ell = 0, 1, \ldots, 2n-1.$$

- Asymptotic expansion of $\boldsymbol{T}_{\boldsymbol{\ell}}$ gives

$$\boldsymbol{T}_\ell = \begin{pmatrix} \eta_\ell \\ \sum_{k=2}^{\infty} c_k h^{k-2} \eta_\ell^k \end{pmatrix}, \quad \ell = 0, 1, \ldots, 2n-1,$$

where c_k depend on y,but not on η_ℓ or h.

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where c_k depend on y,but not on η_ℓ or h.

- More precisely

$$c_k = \frac{2}{k!} \frac{y^{(k)}(0)}{y''(0)}, \quad k = 2, 3, \ldots$$

Solving the nonlinear system

- Our goal is to prove: there exists $h_0 > 0$ such that the system of nonlinear equations has a solution $\boldsymbol{t}$ for any h, $0 \leq h \leq h_0$.

▸ system

Solving the nonlinear system

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  system

- First we find a solution as $h \to 0$.

- Then we prove that the Jacobian matrix in the limit solution is nonsingular.

- Finally, we use the Implicit function theorem.

- The limit solution, as $h \to 0$ is $\boldsymbol{t} = \boldsymbol{\eta} := (\eta_\ell)_{\ell=1}^{2n-2}$.
- Namely

$$\lim_{h\to 0} \sum_{\ell=0}^{n+j} \frac{1}{\dot{\omega}_j(t_\ell)} \boldsymbol{T}_\ell$$

$$= \sum_{\ell=0}^{n+j} \frac{1}{\dot{\omega}_j(\eta_\ell)} \lim_{h\to 0} \boldsymbol{T}_\ell = \sum_{\ell=0}^{n+j} \frac{1}{\dot{\omega}_j(\eta_\ell)} \begin{pmatrix} \eta_\ell \\ \eta_\ell^2 \end{pmatrix}$$

$$= [\eta_0, \eta_1, \ldots, \eta_{n+j}] \begin{pmatrix} \eta \\ \eta^2 \end{pmatrix} = \mathbf{0}.$$

- Unfortunately the Jacobian matrix at the limit solution is singular (its kernel is $n-2$ dimensional).
- The implicit function theorem can not be applied directly!
- Some more involved analysis is needed with several nontrivial steps.
- Finally we end up with the following result.

Theorem

The final system of nonlinear equations has a real solution for $n \leq 5$ and h small enough.

Theorem

If the system of nonlinear equations has a real solution then the interpolating polynomial curve $\boldsymbol{p}_n$ exists and approximates $\boldsymbol{f}$ by optimal approximation order, namely $2n$.

Some particular cases

- In the case $n = 2$ only one equation for a particular unqnown ξ_1 is obtained, i.e.,

$$2\,\xi_1 + c_3 + \mathcal{O}(h) = 0.$$

- It obviously has a real solution.

- If $n = 3$ then the nonlinear system becomes

$$\begin{aligned} \xi_1^2 + 3\,c_3\,\xi_1 + 2\,\xi_2 + c_4 + \mathcal{O}(h) &= 0, \\ 3\,c_3\,\xi_1^2 + 2\,\xi_1\,(\xi_2 + 2\,c_4) + 3\,c_3\,\xi_2 + c_5 + \mathcal{O}(h) &= 0. \end{aligned}$$

- It can be reduced to only one equation for ξ_1

$$\xi_1^3 + \frac{3}{2}\,c_3\,\xi_1^2 + \left(\frac{9}{2}\,c_3^2 - 3\,c_4\right)\xi_1 + \frac{3}{2}\,c_3\,c_4 - c_5 \\ + \mathcal{O}(h) = 0,$$

which again has a real solution.

If $n = 5$ the following "mess" is obtained

$$
\begin{aligned}
c_4 + 5\,c_3\,\xi_1 + 6\,c_2\,{\xi_1}^2 + c_1\,{\xi_1}^3 + 4\,c_2\,\xi_2 + 6\,c_1\,\xi_1\,\xi_2 + \\
{\xi_2}^2 + (3\,c_1 + 2\,\xi_1)\xi_3 + 2\,\xi_4 + \mathcal{O}(h) = 0, \\
c_5 + 6\,c_4\,\xi_1 + 10\,c_3\,{\xi_1}^2 + 4\,c_2\,{\xi_1}^3 + 5\,c_3\,\xi_2 \\
+12\,c_2\,\xi_1\,\xi_2 + 3\,c_1\,{\xi_1}^2\,\xi_2 + \\
3\,c_1\,{\xi_2}^2 + 4\,c_2\,\xi_3 + 6\,c_1\,\xi_1\,\xi_3 \\
+2\,\xi_2\,\xi_3 + 3\,c_1\,\xi_4 + 2\,\xi_1\,\xi_4 + \mathcal{O}(h) = 0,
\end{aligned}
$$

$\longrightarrow$

$$
\begin{aligned}
c_6 + 7\,c_5\,\xi_1 + 15\,c_4\,{\xi_1}^2 + 10\,c_3\,{\xi_1}^3 + 6\,c_4\,\xi_2 + 20\,c_3\,\xi_1\,\xi_2 +\\
12\,c_2\,{\xi_1}^2\,\xi_2 + 6\,c_2\,{\xi_2}^2 + 3\,c_1\,\xi_1\,{\xi_2}^2 + c_2\,{\xi_1}^4 + 5\,c_3\,\xi_3 + 12\,c_2\,\xi_1\,\xi_3 +\\
3\,c_1\,{\xi_1}^2\,\xi_3 + 6\,c_1\,\xi_2\,\xi_3 + {\xi_3}^2 + 4\,c_2\,\xi_4 + 6\,c_1\,\xi_1\,\xi_4 + 2\,\xi_2\,\xi_4 + \mathcal{O}(h) &= 0,\\
c_7 + 8\,c_6\,\xi_1 + 21\,c_5\,{\xi_1}^2 + 20\,c_4\,{\xi_1}^3 + 5\,c_3\,{\xi_1}^4 + 7\,c_5\,\xi_2 + 30\,c_4\,\xi_1\,\xi_2 +\\
30\,c_3\,{\xi_1}^2\,\xi_2 + 4\,c_2\,{\xi_1}^3\,\xi_2 + 10\,c_3\,{\xi_2}^2 + 12\,c_2\,\xi_1\,{\xi_2}^2 + c_1\,{\xi_2}^3 + 6\,c_4\,\xi_3 +\\
20\,c_3\,\xi_1\,\xi_3 + 12\,c_2\,{\xi_1}^2\,\xi_3 + 12\,c_2\,\xi_2\,\xi_3 + 6\,c_1\,\xi_1\,\xi_2\,\xi_3 + 3\,c_1\,{\xi_3}^2 +\\
5\,c_3\,\xi_4 + 12\,c_2\,\xi_1\,\xi_4 + 3\,c_1\,{\xi_1}^2\,\xi_4 + 6\,c_1\,\xi_2\,\xi_4 + 2\,\xi_3\,\xi_4 + \mathcal{O}(h) &= 0.
\end{aligned}
$$

An example

The interpolating curve is

$$\boldsymbol{f}(u) = \begin{pmatrix} \cos u \, \log(1+u) \\ \sin u \, \log(1+u) \end{pmatrix},$$

$u \in [3, 3+h]$. The table shows estimated rate of convergence for the interpolant p_5 on 10 points.

h	Error	Rate
3	$7.12e-6$	–
2.4	$8.79e-7$	9.38
1.92	$1.05e-7$	9.52
1.54	$1.22e-8$	9.63
1.22	$1.40e-9$	9.71
0.98	$1.58e-10$	9.76
0.78	$1.79e-11$	9.77

Nonasymptotic analysis

- Nonasymptotic analysis is much more complicated.
- Geometry of data is involved in the analysis.
- The results are known only for parabolic an cubic case in the plane.
- In higher dimensions it seems that the only known result is interpolation of $d+2$ points by polynomial curve of degree d in $\mathbb{R}^d$.
- Homotopy methods are used to confirm the existence of the solution.

Special curves

- Geometric interpolation of special curves is also interesting (and important).
- Special attention was given to conic sections, specially circular segments.

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- M.S. Floater: An $O(h^{2n})$ Hermite approximation for conic sections. Comput. Aided Geom. Design 14 (1997), no. 2, 135–151.
- G. Jaklič, J. Kozak, M. Krajnc and E. Ž.: On geometric interpolation of circle-like curves. Comput. Aided Geom. Design 24 (2007), no. 5, 241–251.

Theorem

If $x_n(t) := 1 + \sum_{k=2}^{n} \alpha_k\, t^k, \quad y_n(t) := \sum_{k=1}^{n} \beta_k\, t^k, \quad \beta_1 > 0,$ *then the best approximant of the unit circural arc is given by*

$$\alpha_k = \begin{cases} \sum_{j=0}^{k(n-k)} P(j,k,n-k)\cos\left(\frac{k^2}{2n}\pi + \frac{j}{n}\pi\right), & k \text{ is even}, \\ 0, & k \text{ is odd}, \end{cases}$$

$$\beta_k = \begin{cases} 0, & k \text{ is even}, \\ \sum_{j=0}^{k(n-k)} P(j,k,n-k)\sin\left(\frac{k^2}{2n}\pi + \frac{j}{n}\pi\right), & k \text{ is odd}, \end{cases}$$

where $P(j,k,r)$ *denotes the number of integer partitions of* $j \in \mathbb{N}$ *with* $\leq k$ *parts, all between* 1 *and* r*, where* $k, r \in \mathbb{N}$*, and* $P(0,k,r) := 1$*.*

n	$x_n(t), \quad y_n(t)$
2	$x_2(t) = 1 - t^2, \quad y_2(t) = \sqrt{2}\, t$
3	$x_3(t) = 1 - 2\,t^2, \quad y_3(t) = 2\,t - t^3$
4	$x_4(t) = 1 - (2+\sqrt{2})t^2 + t^4$ $y_4(t) = \sqrt{4+2\sqrt{2}}(t - t^3)$
5	$x_5(t) = 1 - (3+\sqrt{5})t^2 + (1+\sqrt{5})t^4$ $y_5(t) = (1+\sqrt{5})t - (3+\sqrt{5})t^3 + t^5$
6	$x_6(t) = 1 - 2(2+\sqrt{3})t^2 + 2(2+\sqrt{3})t^4 - t^6$ $y_6(t) = (\sqrt{2}+\sqrt{6})t - \sqrt{2}\,(3+2\sqrt{3})t^3 + (\sqrt{2}+\sqrt{6})t^5$

Table : The best approximats from the previous Theorem.

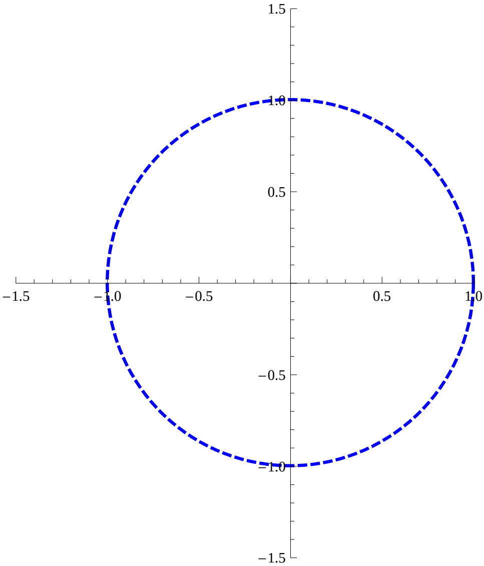

Figure : The unit circle.

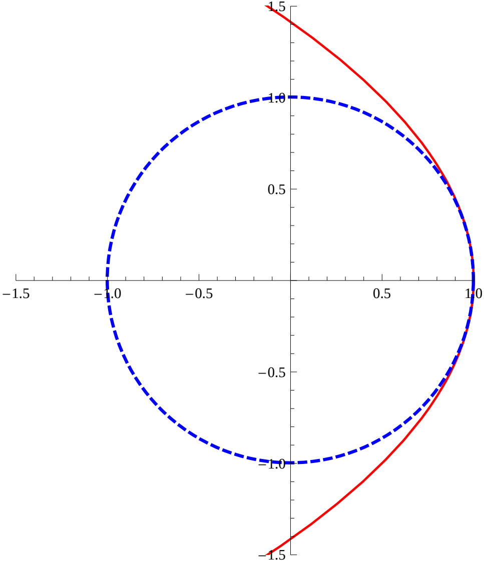

Figure : The unit circle and its polynomial approximant for $n = 2$.

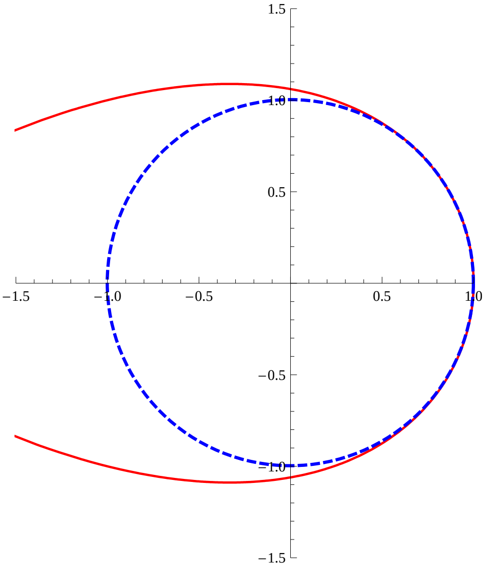

Figure : The unit circle and its polynomial approximant for $n = 3$.

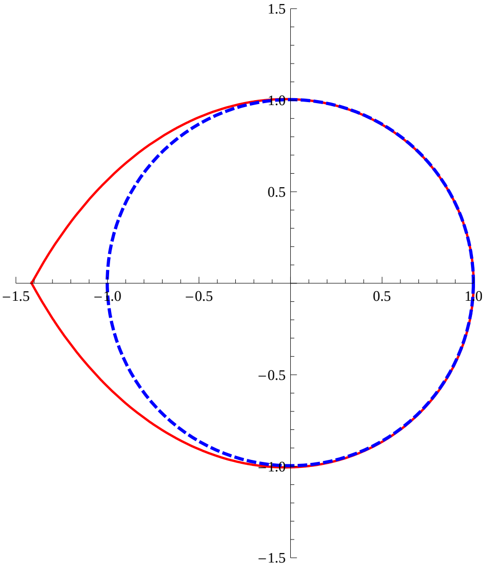

Figure : The unit circle and its polynomial approximant for $n = 4$.

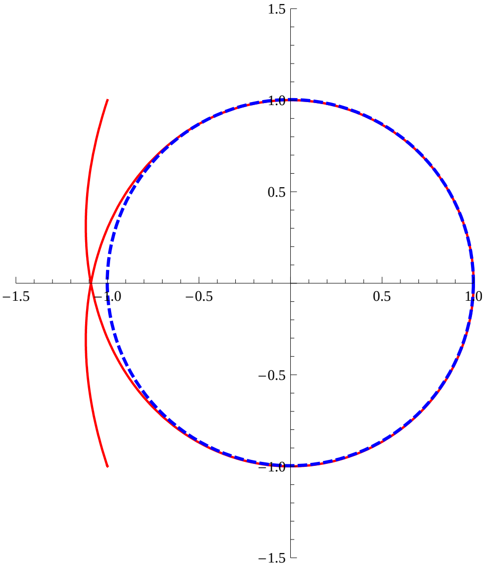

Figure : The unit circle and its polynomial approximant for $n = 5$.

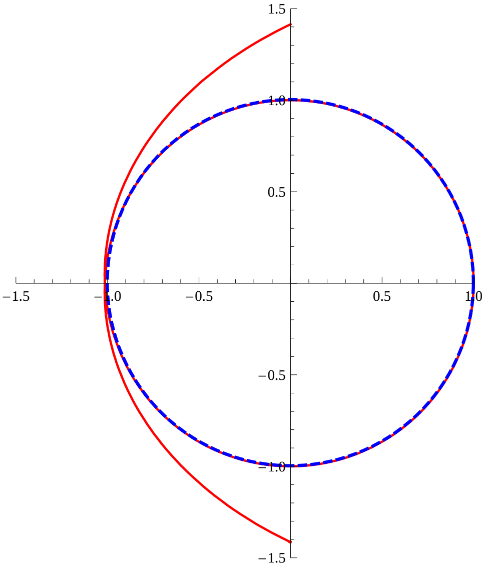

Figure : The unit circle and its polynomial approximant for $n = 6$.

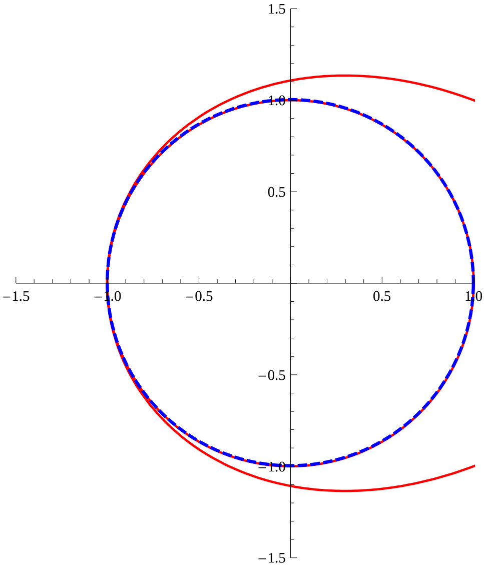

Figure : The unit circle and its polynomial approximant for $n = 7$.

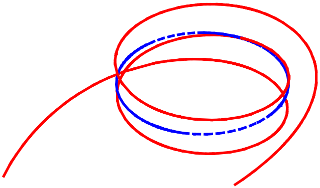

Figure : Cycles of the approximant for $n = 20$.

Open problems

- Asymptotic analysis for $n > 5$.
- Geometric conditions implying solutions at least for $n \leq 5$.
- Geometric interpolation of special classes of curves (PH curves, MPH curves,...) (partially solved).
- Geometric interpolation of spatial and rational curves (connected with motion design (robotics)).
- Geometric subdivision.